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ASYMPTOTIC FREEDOM FROM THERMAL AND VACUUM MAGNETIZATION

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Abstract

We calculate the effective Lagrangian for a magnetic field in spinor, scalar and vector QED. Connections are then made to $SU(N_C)$ Yang–Mills theory and QCD. The magnetization and the corresponding effective charge are obtained from the effective Lagrangian. The renormalized vacuum magnetization will depend on the renormalization scale chosen. Regardless of this, the effective charge decreasing with the magnetic field, as in QCD, corresponds to anti- screening and asymptotic freedom. In spinor and scalar QED on the other hand, the effective charge is increasing with the magnetic field, corresponding to screening. Including effects due to finite temperature and density, we comment on the effective charge in a degenerate fermion gas, increasing linearly with the chemical potential. Neglecting the tachyonic mode, we find that in hot QCD the effective charge is *decreasing* as the inverse temperature, in favor for the formation of a quark-gluon plasma. However, including the real part of the contribution from the tachyonic mode, we find instead an effective charge *increasing* with the temperature. Including a thermal gluon mass, the effective charge in hot QCD is group invariant (unlike in the two cases above), and decreases logarithmically in accordance to the vacuum renormalization group equation, with the temperature as the momentum scale.

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1 Introduction

We calculate the one loop effective Lagrangian density for a static uniform magnetic field in different gauge theories. From this the effective charge, the magnetization, and other relevant objects are obtained. In a recent publication [1], the vacuum of spinor and scalar QED was shown to exhibit paramagnetic properties. On the other hand, Nielsen related in a well known article [2], asymptotic freedom of QCD to a paramagnetic vacuum of mass-less QCD. With the same reasoning, the vacuum of Abelian gauge theories (like QED) should exhibit a diamagnetic behavior. In the present letter we resolve the ostensible discrepancy between the two different approaches above. This discrepancy originates in the use of renormalized quantities in Ref. [1], verses the use of bare quantities, in order to get analogy with a classical dielectricum, in Ref. [2]. In Section 2 we shall briefly review the properties of a dielectricum, and relate the effective charge to the magnetic behavior. In Section 3 we consider the generic effective Lagrangian for a background magnetic field, and how the magnetization is obtained from it. We also give another definition of an effective charge, applicable also at finite temperature and density. In Section 4 we calculate the vacuum effective Lagrangian for spinor, scalar and vector QED in the massive as well as the mass-less case. Similar calculations have earlier been performed in for example Ref. [3] (in terms of the vacuum energy), and in Ref. [4] (in QED).

Recently much attention has been paid to the β-function and the effective charge in QCD at high temperature (see e.g. Refs. [5, 6, 7, 8, 9]), and high density (see e.g. Ref. [10]), motivated by the suggested formation of a quark–gluon plasma under such circumstances [11]. The results differ depending on the different choices of gauge, gauge fixing parameter [7, 8], and on the vertex considered [5]. The effective charge obtained from the effective Lagrangian of a background field should be free of these diseases. In the background field formalism, as used also in Refs. [6, 7, 8, 9], the renormalized couplings of the different vertices are kept equal and related to the renormalization of the background field. We shall in Section 5 calculate the effective Lagrangian in the presence of a heat and charge bath, and relate it to the free energy of the plasma. Summing only over the physical degrees of freedom, this should be gauge independent. However, to perform the explicit calculations we have chosen a certain gauge for the background magnetic field, and for the mass-less gluon field. Effective Lagrangians for a static uniform magnetic field in a thermal environment have been considered earlier. For scalar and spinor QED in for example Refs. [12, 13], and in $SU(N_c)$ Yang–Mills theories (for $N_c = 2, 3$) in for example Refs. [14, 15, 16, 17, 18, 19]. However, in the Yang–Mills theories, they focused on the effective Lagrangian (or equivalently the thermodynamic pressure) in order to investigate a possible phase transition, and we disagree with some of the previous results. Since perturbation theory in this situation actually only is valid after the phase transition to a

quark–gluon plasma has taken place, we shall here mainly focus on the effective charge obtained from the effective Lagrangian. A corresponding effective charge has earlier been considered in QED [20, 13], but to our knowledge not in non-Abelian gauge theories.

We shall use a naive real-time formalism, valid here since there are no propagators with coinciding momenta, and calculate the thermal effective Lagrangian in spinor, scalar and vector QED. $SU(N_c)$ Yang-Mills theory is then related to charged mass-less vector bosons, and connections are made with QCD in Section 6. When possible we compare with previous results, and make some corrections. We also consider the effects when a thermal gluon mass is taken into account. Finally we discuss the results here obtained, and their relevance in Section 7.

Since the topics here considered provide an extremely nice and simple example of renormalization, we shall be fairly explicit.

2 Magnetization and (Anti-) Screening

In his very pedagogical example, Nielsen[2] described asymptotic freedom (anti-screening) in QCD, in the same way as the intuitively clear picture of screening in an ordinary dielectric medium. Let us first recapitulate some of the basic features of a classical homogeneous dielectricum. The effective coupling of two test charges separated by a distance r is

$$e_{\varepsilon}^2(r) = \frac{e_0^2}{\varepsilon(r)} \quad , \tag{2.1}$$

where e_0 is the undressed charge measured in the absence of the medium. The dielectric permittivity $\varepsilon(r)$ must approach unity as $r \to 0$, since then there is no shielding medium between the two test charges. Screening means that the medium will be polarized around the test charge, so that the effective charge measured at $r > 0$, will be smaller than the undressed charge (at $r = 0$). In terms of the dielectric permittivity we thus have $\varepsilon > 1$. If, on the other hand, the polarization of the medium is such that the effective charge is larger than the undressed charge we have anti-screening, and correspondingly $\varepsilon < 1$.

In analogy to this, Nielsen calculated the effective charge in QCD, with the dielectricum consisting of the quantum mechanical vacuum, containing virtual particles only. In a quantum field theory one will encounter divergences that after a proper regularization may be removed by a rescaling of the parameters appearing in the original Lagrangian (renormalization). In a classical theory there are no such divergences, so in order to get full analogy with the example of a dielectric medium, Nielsen used cut-off regularized bare (i.e. before renormalization) quantities. An ultraviolet momentum cut-off Λ, corresponds to a smallest distance $r_0 = 1/\Lambda$, where the bare charge is measured, corresponding to the classical undressed charge, at $r = 0$.

The relativistic invariance of the vacuum of a quantum field theory requires that the permittivity is connected to the magnetic permeability $\mu_{\rm perm}$, through

$$\varepsilon\mu_{\rm perm} = 1 \quad . \tag{2.2}$$

This has no counterpart in an ordinary polarizable medium. It turns out that it is easier to calculate the magnetic susceptibility χ (such that $\mu_{\rm perm} = 1 + \chi$) in a background (color) magnetic field B, than to directly calculate the permittivity in a (color) electric field. By a heuristic reasoning, Nielsen related the distance to the field strength, according to $r \approx 1/\sqrt{eB}$ (we are assuming $eB > 0$). We thus find the dielectric permittivity

$$\varepsilon(r) = \left.\frac{1}{1+\chi(B)}\right|_{eB\to 1/r^2} \quad . \tag{2.3}$$

In terms of bare regularized quantities (denoted by the subscript "b"), we thus find that screening ($\varepsilon_b > 1$), corresponds to $\chi_b < 0$, i.e. diamagnetism; and anti-screening ($\varepsilon_b < 1$), corresponds to $\chi_b > 0$, i.e. paramagnetism.

How could then possibly the Abelian gauge theories discussed in Ref. [1] exhibit a paramagnetic behavior, when they cannot be asymptotically free?

The answer to this lies in the renormalization procedure. We can never measure such things as bare quantities, but renormalization is necessary in order to obtain the physical parameters. The renormalization is performed at some momentum scale λ, arbitrary but finite. The reference charge will be the renormalized charge, measured at momentum scale λ, or equivalently on a distance $r_1 \approx 1/\lambda$. For the clarity of the reasoning, and the comparison with the classical dielectricum, we are assuming some sort of on–shell renormalization, where the renormalized charge corresponds to the charge measured at the renormalization scale. In general it is sufficient only to remove the divergences in the renormalization procedure. Then one will have some finite relation between the physical charge actually measured, and the renormalized charge that just is a parameter of the theory. In analogy to the classical dielectricum we thus have

$$e_\varepsilon^2(r) = \frac{e_1^2}{\varepsilon_1(r)} \quad , \tag{2.4}$$

where e_1^2 is the charge measured at distance r_1, i.e. $\varepsilon_1(r_1) = 1$, corresponding to the charge renormalized at momentum $\lambda_1 \approx 1/r_1$. The general criteria for screening is that the effective charge should decrease with the distance or increase with the momentum scale, and vice versa for anti-screening. When considering distances smaller than r_1, screening instead corresponds to $\varepsilon_1 < 1$, and anti-screening to $\varepsilon_1 > 1$. Without knowledge of the scale at which the reference charge is measured, the magnitude of ε, will thus not tell whether we have screening or anti-screening.

In the renormalized spinor and scalar QED considered in Ref. [1] the renormalization is performed at the lowest momentum scale (as is standard), corresponding to $r_1 \to \infty$. In this case we thus always have $r < r_1$, and screening here corresponds to $\varepsilon < 1$, which means paramagnetism $\chi > 0$, and similarly anti-screening would correspond to $\varepsilon > 1$, and thereby $\chi < 0$, i.e. diamagnetism. However, if the theory exhibits anti-screening (like QCD), the effective charge is becoming infinitely large at large distances, so the renormalization cannot be performed at vanishing momentum scale.

3 Effective Lagrangian and Effective Coupling

The generic effective Lagrangian for a background magnetic field may be separated into its different contributions

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{tree}} + \mathcal{L}_{\text{vac}} + \mathcal{L}_{\text{mat}} \quad . \tag{3.1}$$

Here

$$\mathcal{L}_{tree} = -B^2/2 \qquad , \tag{3.2}$$

is the free tree level part; $\mathcal{L}_{\text{vac}}$ is the vacuum contribution due to virtual particles; and $\mathcal{L}_{\text{mat}}$ is the contribution due to real particles at finite temperature and density. The vacuum contribution $\mathcal{L}_{\text{vac}}$ is calculated to one-loop order in Section 4, and the matter contribution $\mathcal{L}_{\text{mat}}$ will be considered in Section 5.

From the effective Lagrangian we may obtain the magnetization

$$M \equiv M_{\text{vac}} + M_{\text{mat}} = \frac{\partial}{\partial B}(\mathcal{L}_{\text{vac}} + \mathcal{L}_{\text{mat}}) \quad . \tag{3.3}$$

The vacuum magnetization, originally proposed in Ref. [1], is a real physical quantity, but only gives measurable effects at extremely high field strengths. Performing another derivative we find the magnetic susceptibility

$$\chi \equiv \chi_{\text{vac}} + \chi_{\text{mat}} = \frac{\partial}{\partial B}(M_{\text{vac}} + M_{\text{mat}}) \quad . \tag{3.4}$$

At vanishing temperature and density we may use the vacuum magnetic susceptibility to obtain the effective charge according to Eq. (2.3) and Eq. (2.4). In the presence of matter at finite temperature (T) and chemical potential (μ), Lorentz invariance is broken when choosing a preferred frame of reference in which the medium is at rest and in equilibrium. Then there is no connection between the permittivity and the permeability. We must therefore find another way of obtaining the effective charge in this case. It may be done through the identification [20]

$$\mathcal{L}_{\text{eff}} \approx -\frac{1}{2}B_{\text{eff}}^2 = -\frac{1}{2}\frac{(eB)^2}{e_{\text{eff}}^2} \quad , \tag{3.5}$$

where we have used that $eB = e_{\rm eff} B_{\rm eff}$ is required to be invariant under renormalization. Performing the derivative with respect to $(eB)^2$, that is invariant under renormalization as well as Lorentz transformations ($B^2 - E^2$ is Lorentz invariant and here reduced to B^2), we find

$$\frac{1}{e_{\rm eff}} \equiv -2\frac{\partial \mathcal{L}_{\rm eff}}{\partial (eB)^2} = \frac{1}{e^2} - \frac{1}{eB}\frac{M}{e} \quad , \tag{3.6}$$

where M is the magnetization. Notice that one may also define the magnetic susceptibility as the response function of the magnetization $M = \chi B$. In this case we find

$$\frac{1}{e_\varepsilon^2} = \frac{1}{e^2}\frac{1}{1+\chi} \simeq \frac{1}{e^2} - \frac{1}{eB}\frac{M}{e} \quad , \tag{3.7}$$

to the lowest order in the coupling.

An external magnetic field H is introduced by adding a term $\mathcal{L}_{\rm ext} = j^\nu_{\rm ext} A_\nu$ to the effective Lagrangian. Neglecting a surface term we find $\mathcal{L}_{\rm ext} = \mathbf{B} \cdot \mathbf{H}$. By construction $\mathcal{L}_{\rm eff}$ is invariant under renormalization. If $\mathcal{L}_{\rm eff} + \mathcal{L}_{\rm ext}$ is to be invariant, the invariance of eB requires H/e to be invariant. This seems quite reasonable, since $\nabla \times \mathbf{H} = \mathbf{j}_{\rm ext}$. The external field H is thus only a function of the external charges, and thus should scale as a charge. Minimizing $\mathcal{L}_{\rm eff}$ with respect to B we find the mean-field equation

$$B = H + M_{\rm vac}(B) + M_{\rm mat}(B) \quad . \tag{3.8}$$

This equation is telling us how to find the average microscopic field B, that also is the acting field felt by the particles in the medium, in the presence of an external field H.

4 The Vacuum of Spinor, Scalar and Vector QED

We shall here consider the one-loop vacuum contribution to the effective Lagrangian of spinor, scalar and vector QED with an external static uniform magnetic field $\mathbf{B} = B\mathbf{e}_z$. In order to ease the physical interpretation, and the comparison with Ref. [2], we shall use a cut-off regularization procedure throughout this Section.

The Lagrangian for a spin 1/2 particle of charge $-e$ coupled to the electro-magnetic field is

$$\mathcal{L}^{\rm ferm} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\not{D} - m)\psi \quad , \tag{4.1}$$

where the covariant derivative is $D_\nu = \partial_\nu - ieA_\nu$, and we have used the shorthand notation $\not{D} \equiv \gamma^\mu D_\mu$. The corresponding Lagrangian for the spin 0 case reads

$$\mathcal{L}^{\rm scal} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D^*_\nu \phi^\dagger D^\nu \phi - m^2 \phi^\dagger \phi \quad . \tag{4.2}$$

For reasons that will become obvious in section 6 we are also interested in the case of a spin 1 vector particle with gyro-magnetic ratio $\gamma = 2$. The corresponding Lagrangian is then obtained by adding an anomalous magnetic moment term to the minimally coupled Lagrangian [21], with the result

$$\begin{aligned} \mathcal{L}^{\text{vec}} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(D^*_\mu W^\dagger_\nu - D^*_\nu W^\dagger_\mu)(D^\mu W^\nu - D^\nu W^\mu) + m^2 W^\dagger_\mu W^\mu \\ &\quad -ie(\gamma-1)F^{\mu\nu}\frac{W^\dagger_\mu W_\nu - W_\mu W^\dagger_\nu}{2} \quad . \end{aligned} \tag{4.3}$$

The general quantized theory described by $\mathcal{L}^{\text{vec}}$ is not even renormalizable [22]. But here we shall consider no virtual photons. The renormalizability then follows in exactly the same way as for scalar and spinor QED. We separate the vector-potential into $A_\nu + \tilde{A}_\nu$, where A_ν is the vector-potential for $\mathbf{B}$, and $\tilde{A}_\nu$ corresponds to the quantized radiation field. We shall here only calculate the one-loop effective Lagrangian. We may then neglect the radiation field, apart from the kinetic term $-\tilde{F}^2/2$. For simplicity we shall not write out the radiation field, but only add the contribution from thermal photons (or gluons) in the end.

A particle of charge $-e$, spin $\boldsymbol{\sigma}$ and gyro-magnetic ratio γ, has the spin magnetic moment $\boldsymbol{\mu} = -\gamma e\langle\boldsymbol{\sigma}\rangle/2m$. With spin projection s along $-e\mathbf{B}$ the energy spectrum reads [3]

$$E_{n,s} = \sqrt{m^2 + p_z^2 + eB(2n+1-\gamma s)} \quad , \tag{4.4}$$

where the dependence on the momentum in the direction of the field (p_z) has been suppressed, and we assume $eB > 0$.

It is illustrated how to calculate the vacuum contribution at one–loop level in spinor QED in Refs. [4, 13], and also in scalar QED in Ref. [13]. In the general case we find similarly, using the trick to perform the derivative with respect to the mass

$$\frac{\partial \mathcal{L}_{\text{vac,b}}}{\partial m^2} = i(-1)^{2\sigma+1}\frac{eB}{(2\pi)^3}\sum_s\sum_{n=0}^{\infty}\int d\omega dp_z \frac{1}{w^2 - E_{n,s}^2 + i\varepsilon} \quad , \tag{4.5}$$

where σ is the spin of the intermediate particles. We have here suppressed that the contribution at $B = 0$ should be subtracted, due to the normalization of the generating functional. The subscript b denotes bare quantities, i.e. before renormalization. The difference in sign comes from the fermionic Grassmann algebra.

We may now use Cauchy's theorem to perform the integration over ω. Integrating with respect to m^2 we find, with the subtraction explicit

$$\mathcal{L}_{\text{vac,b}} = (-1)^{2\sigma+1}\frac{eB}{(2\pi)^2}\sum_s\sum_{n=0}^{\infty}\int dp_z E_{n,s} - \mathcal{L}_{\text{vac,b}}(B=0) \quad . \tag{4.6}$$

In order to interpret this expression we shall calculate the density of states for particles in a box $V = L_x L_y L_z$. We then need that the wave functions in each of the theories here

considered, with the choice of gauge $A_\nu = (0,0,Bx,0)$, are of the generic form

$$\Psi^{(\pm)}_{n,p_y,p_z,s}(\mathbf{x},t) = \frac{1}{2\pi\sqrt{2E_{n,s}}} \exp[\pm i(-E_{n,s}t+p_y y+p_z z)]\, I_{n;p_y}(x) \quad , \tag{4.7}$$

$$I_{n;p_y}(x) \equiv \left(\frac{eB}{\pi}\right)^{1/4} \exp\left[-\frac{1}{2}eB\left(x-\frac{p_y}{eB}\right)^2\right] \frac{1}{\sqrt{n!}} H_n\left[\sqrt{2eB}\left(x-\frac{p_y}{eB}\right)\right] , \tag{4.8}$$

where H_n is a Hermite polynomial. Without the magnetic field, the number of states on the momentum interval dp_i is $L_i/2\pi\, dp_i$. With the above magnetic field, this still holds true for $i = z$. The wave-functions in Eq. (4.8) corresponds to a harmonic oscillator, centered at $x_0 = p_y/eB$, that has to be within the box, i.e. $0 \leq p_y/eB \leq L_x$. Since the energy is independent of p_y we must sum over all possible values. The resulting degeneracy of states is then

$$V\frac{eB}{(2\pi)^2} \quad . \tag{4.9}$$

The vacuum Lagrangian density is thus equal to the negative vacuum energy density

$$\mathcal{L}_{\text{vac,b}} = \frac{-1}{V}[E_{\text{vac}} - E_{\text{vac}}(B=0)] \quad , \tag{4.10}$$

again without the contribution at vanishing magnetic field. The vacuum energy was the starting point in Ref. [2]. The vacuum energy needs to be regularized. As it stands it is quadratically divergent, but when the $B = 0$ part has been subtracted the result is only logarithmically divergent.

Instead of immediately integrating with respect to m^2, we shall here first use

$$\frac{1}{2E} = \frac{1}{\sqrt{\pi}} \int_{1/\Lambda}^{\infty} \exp[-E^2x^2]dx + \mathcal{O}(\frac{1}{\Lambda}) \quad , \tag{4.11}$$

where we have introduced the ultra-violet cut-off Λ, that essentially removes the contributions for $E > \Lambda$ due to the exponential suppression. We may now perform the Gaussian integral over p_z, sum the infinite geometrical series in n, and integrate with respect to m^2. Subtracting the contribution for $B = 0$, and changing variable of integration to $t = x^2$, we arrive at

$$\mathcal{L}_{\text{vac,b}} = \frac{(-1)^{2\sigma}}{16\pi^2} \sum_s \int_{1/\Lambda^2}^{\infty} \frac{dt}{t^3} \exp(-m^2t) \left\{ \frac{eBt}{\sinh(eBt)} \exp(eBt\gamma s) - 1 \right\} \quad . \tag{4.12}$$

For $eB(\gamma s - 1) > m^2$ this is divergent for large t. This is the case for large fields in vector QED ($\sigma = 1$) with $\gamma = 2$, that will be treated separately below. Let us now first consider the massive case in Section 4.1, and then the mass-less case in Section 4.2.

4.1 The Massive Case

In the limit as $x \equiv eBt \to 0$, we have

$$\frac{x}{\sinh(x)} \exp(\gamma s x) = 1 + \gamma s x + \frac{1}{2}x^2\left(\gamma^2 s^2 - \frac{1}{3}\right) + \mathcal{O}(x^3) \quad . \tag{4.13}$$

Notice that $\sum_s s = 0$. Define from the quadratic term

$$\hat{\chi} \equiv \frac{(-1)^{2\sigma}}{16\pi^2} \sum_s \left(\gamma^2 s^2 - \frac{1}{3} \right) \quad . \tag{4.14}$$

Let us now subtract and add the term containing

$$\int_{1/\Lambda^2}^{\infty} \frac{dt}{t} \exp(-m^2 t) = E_1(m^2/\Lambda^2) = -\gamma_E + \ln\left(\frac{\Lambda^2}{m^2}\right) + \mathcal{O}\left(\frac{m^2}{\Lambda^2}\right) \quad , \tag{4.15}$$

where E_1 is an exponential integral, and $\gamma_E = 0.57721566\ldots$ is Euler's constant. In order to get rid of the cutoff Λ we must now perform a renormalization. The Ward identities of the theories here considered requires the product eB to be invariant. Let us therefore rescale the external field and the coupling according to

$$\begin{aligned} e_b^2 &= Z_\lambda^{-1} e^2(\lambda) \quad , & (4.16) \\ B_b &= Z_\lambda^{1/2} B(\lambda) \quad , & (4.17) \end{aligned}$$

where $e(\lambda)$ and $B(\lambda)$ are the charge and background field renormalized at momentum scale λ, respectively. Adding $0 = \ln(\lambda^2/\lambda^2)$, we may write

$$\mathcal{L}_{\text{eff}} = -\frac{1}{2}\frac{(eB)^2}{e^2(\lambda)} - \frac{1}{2}(Z_\lambda - 1)\frac{(eB)^2}{e^2(\lambda)} + \frac{1}{2}(eB)^2\hat{\chi}\left[-\gamma_E + \ln\left(\frac{\Lambda^2}{\lambda^2}\right)\right] + \mathcal{L}_{\text{vac}}(eB,\lambda) \quad , \tag{4.18}$$

where the finite, renormalized vacuum contribution is

$$\begin{aligned} \mathcal{L}_{\text{vac}}(eB,\lambda) &= \frac{1}{2}(eB)^2\hat{\chi}\ln\left(\frac{\lambda^2}{m^2}\right) + \tilde{\mathcal{L}}_{\text{vac}}(eB) \quad , & (4.19) \\ \tilde{\mathcal{L}}_{\text{vac}}(eB) &= \int_0^{\infty} \frac{dt}{t^3} \exp(-m^2 t) \left\{ \frac{(-1)^{2\sigma}}{16\pi^2} \sum_s \left[\exp(eBt\gamma s) \frac{eBt}{\sinh(eBt)} - 1 \right] - \hat{\chi}\frac{(eBt)^2}{2} \right\} \quad . & (4.20) \end{aligned}$$

We must now choose Z_λ in such a way that the divergence as $\Lambda \to \infty$ is removed

$$Z_\lambda - 1 = e^2(\lambda^2)\hat{\chi}\left[-\gamma_E + \ln\left(\frac{\Lambda^2}{\lambda^2}\right)\right] \quad , \tag{4.21}$$

i.e.

$$\frac{1}{e^2(\lambda)} = \frac{1}{e_b^2} + \hat{\chi}\left[-\gamma_E - \ln\left(\frac{\Lambda^2}{\lambda^2}\right)\right] \quad . \tag{4.22}$$

Performing the derivative with respect to λ we find the lowest order β-function

$$\lambda\frac{de(\lambda)}{d\lambda} \equiv \beta[e(\lambda)] = -\hat{\chi}e^3(\lambda) \quad . \tag{4.23}$$

In the case of (spinor) QED, $\sigma = 1/2$ and $\gamma = 2$, we find

$$\hat{\chi}^{\rm ferm} = -\frac{1}{12\pi^2} \quad . \tag{4.24}$$

Similarly in scalar QED, $\sigma = 0$ gives

$$\hat{\chi}^{\rm scal} = -\frac{1}{48\pi^2} \quad . \tag{4.25}$$

Inserting this into Eq. (4.23), we recognize the correct β-function of spinor and scalar QED. In the theory of vector QED with gyro-magnetic ratio $\gamma = 2$, the β-function really does not exist, due to the lack of renormalizability. However, for vector bosons interacting only with the external field we find

$$\hat{\chi}^{\rm vec}|_{\rm massive} = \frac{7}{16\pi^2} \quad . \tag{4.26}$$

This does not agree with the renormalization of the coupling considered in Ref. [22] (corresponding to $\hat{\chi}^{\rm vec} = 5/16\pi^2$). Notice the crucial difference in sign of $\hat{\chi}$ for vector QED, that we later shall relate to QCD. The solution to the above renormalization group equation may be written

$$\frac{1}{e^2(\lambda)} = \frac{1}{e^2(\lambda_0)} + \hat{\chi} \ln\left(\frac{\lambda^2}{\lambda_0^2}\right) \quad . \tag{4.27}$$

With $\hat{\chi} < 0$ this corresponds to the effective charge *increasing* with the momentum scale λ, whereas $\hat{\chi} > 0$ corresponds to the effective charge *decreasing*. Obviously $e(\lambda)$ and $B(\lambda)$ are defined such that $\mathcal{L}_{\rm eff}$ is independent of λ. We have

$$\mathcal{L}_{\rm eff} = -\frac{1}{2}\frac{(eB)^2}{e^2(\lambda)} + \frac{1}{2}(eB)^2\hat{\chi} \ln\left(\frac{\lambda^2}{m^2}\right) + \tilde{\mathcal{L}}_{\rm vac}(eB) \quad . \tag{4.28}$$

Notice that Z_λ only depends on λ through the dimension-less parameter λ/Λ. As long as the Λ dependence is absorbed in Z_λ (as is necessary to get finite expressions) the RGE(4.23) will not depend on how Z_λ and thus $e(\lambda)$ are defined. Since the cutoff Λ no longer appears in the effective Lagrangian, we may now send it back to infinity where it belongs.

In order to obtain a physically clear picture let us now consider the high field limit. Substitute $y = eBt$ in Eq. (4.20), and split the integral at y_0, such that $eB/m^2 \gg y_0 \gg 1$. The contribution from $y < y_0$ is easily seen to be $\mathcal{O}[(eB)^2]$. The contribution from $y > y_0$ is dominated by the subtracted term $\hat{\chi}(eBt)^2/2$, that gives

$$\frac{\tilde{\mathcal{L}}_{\rm vac}}{(eB)^2} \simeq -\frac{1}{2}\hat{\chi}\frac{m^2}{(eB)^2}\int_{y_0}^{\infty}\frac{dy}{y}\exp\left(-\frac{m^2}{eB}y\right) \simeq -\frac{1}{2}\hat{\chi}\ln\left(\frac{eB}{m^2}\right) \quad . \tag{4.29}$$

At the energy scale $eB = \lambda^2 \gg m^2$, the two logarithms will cancel, and thus the effective Lagrangian assumes a purely Maxwellian form

$$\mathcal{L}_{\rm eff} \simeq -\frac{1}{2}B^2(\lambda^2 = eB) \quad , \; eB \gg m^2 \quad . \tag{4.30}$$

We thus find that at least in the limit $\lambda \gg m$, that $e(\lambda)$ and $B(\lambda)$ are the charge and field strength measured at momentum scale $eB = \lambda^2$, respectively.

In the high field limit ($eB \gg m^2$), Eq. (4.29) gives to leading order

$$
\begin{aligned}
\frac{M_{\rm vac}(\lambda)}{e(\lambda)} &= -eB\,\hat{\chi}\,\ln\!\left(\frac{eB}{\lambda^2}\right) \quad , && (4.31)\\
\chi_{\rm vac}(\lambda) &= -e^2(\lambda)\,\hat{\chi}\,\ln\!\left(\frac{eB}{\lambda^2}\right) \quad . && (4.32)
\end{aligned}
$$

We see that the sign of the magnetic susceptibility in addition to $\hat{\chi}$ depends on the relative magnitude between eB and λ^2. This does anyhow correspond to screening in spinor and scalar QED ($\hat{\chi} < 0$); but anti-screening in vector QED ($\hat{\chi} > 0$) since

$$
\frac{\partial \chi_{\rm vac}(\lambda)}{\partial (eB)} = -e^2(\lambda)\hat{\chi}\,\frac{1}{eB} \quad . \tag{4.33}
$$

The permittivity ε is thus decreasing with the energy scale eB in spinor and scalar QED (with $\hat{\chi} < 0$) , but increasing in vector QED (with $\hat{\chi} > 0$).

In the high field limit we find the effective charge

$$
\frac{1}{e^2_{\rm eff}(eB)} \simeq \frac{1}{e^2(\lambda_0)} + \hat{\chi}\ln\!\left(\frac{eB}{\lambda_0^2}\right) \quad , \; eB \gg m^2 \quad , \tag{4.34}
$$

that is a solution to the lowest order RGE with the momentum scale identified as $\lambda = \sqrt{eB}$. Notice that $e_{\rm eff}$ must be independent of λ_0 by its definition in Eq. (3.6), since $\mathcal{L}_{\rm eff}$ and eB so is, and that this follows from Eq. (4.27). It is no coincidence that the above effective coupling satisfies the lowest order RGE, since it corresponds to a summation of the leading logarithms. The high field limit is dominated by the term containing $\hat{\chi}$, subtracted in the renormalization, that is defining the β–function.

The effective Lagrangian will assume a particularly simple form if we choose $\lambda = m$

$$
\mathcal{L}_{\rm eff} = -\frac{1}{2}B^2 + \tilde{\mathcal{L}}_{\rm vac}(eB) \quad . \tag{4.35}
$$

Here we have suppressed the dependence on λ, since in the weak field limit

$$
\mathcal{L}_{\rm eff} = -\frac{1}{2}B^2 + B^2\mathcal{O}\left[e^2\frac{(eB)^2}{m^4}\right] \quad , \tag{4.36}
$$

so that e and B are the ordinary electric charge and magnetic field, measured at vanishingly small magnetic field, corresponding to long wavelengths. The vacuum of spinor and scalar QED will thus be paramagnetic ($\chi_{\rm vac} > 0$) in the high field limit $\{eB \gg \lambda^2 = m^2\}$. Actually, in the spinor case it is paramagnetic for all values of the magnetic field, as shown in Ref. [1], to which we refer for a more extensive treatment of the vacuum magnetization

in spinor and scalar QED. In spinor QED we have the vacuum contribution to the effective Lagrangian explicitly

$$\mathcal{L}_{\rm vac}^{\rm ferm} = -\frac{1}{8\pi^2}\int_0^\infty \frac{dt}{t^3}\exp(-m^2 t)\left[eBt\coth(eBt) - 1 - \frac{1}{3}(eBt)^2\right] \quad . \tag{4.37}$$

In scalar QED we find

$$\mathcal{L}_{\rm vac}^{\rm scal} = \frac{1}{16\pi^2}\int_0^\infty \frac{dt}{t^3}\exp(-m^2 t)\left[\frac{eBt}{\sinh(eBt)} - 1 + \frac{1}{6}(eBt)^2\right] \quad . \tag{4.38}$$

As remarked above the case of vector QED for $eB > m^2$ needs more careful considerations. Let us use the similarity between vector and scalar QED to write

$$\mathcal{L}_{\rm vac,b}^{\rm vec}\Big|_{\rm massive} = 3\mathcal{L}_{\rm vac,b}^{\rm scal} + \Delta\mathcal{L}_{\rm vac,b}^{\rm vec} \quad . \tag{4.39}$$

Written on the vacuum energy form corresponding to Eq. (4.6) the extra term reads

$$\Delta\mathcal{L}_{\rm vac,b}^{\rm vec} = -\frac{eB}{(2\pi)^2}\int dp_z\left(\sqrt{m^2 + p_z^2 - eB - i\varepsilon} - \sqrt{m^2 + p_z^2 + eB}\ \right) \quad . \tag{4.40}$$

Obviously, this will contain an imaginary part for $eB > m^2$

$$\mathrm{Im}\,\Delta\mathcal{L}_{\rm vac}^{\rm vec} = \frac{eB}{2\pi^2}\int_{-\sqrt{eB-m^2}}^{\sqrt{eB-m^2}} dp_z\sqrt{eB - m^2 - p_z^2} = \Theta(eB - m^2)\frac{(eB - m^2)eB}{8\pi} \quad . \tag{4.41}$$

Working out the real part in analogy to the general case we find

$$\begin{aligned}\Delta\mathcal{L}_{\rm vac,b}^{\rm vec} \quad = \quad & \frac{1}{2}\frac{(eB)^2}{2\pi^2}\ln\left(\frac{\Lambda^2}{\sqrt{(m^2+eB)|m^2 - eB|}}\right) - \frac{eBm^2}{8\pi^2}\left[\ln\left(\frac{m^2+eB}{|m^2 - eB|}\right) - 2\frac{eB}{m^2}\right] \\ & -C(eB)^2 + i\Theta(eB - m^2)\frac{(eB - m^2)eB}{8\pi} \quad ,\end{aligned} \tag{4.42}$$

where C is an irrelevant constant that anyhow will be renormalized away. Notice that the result also may be obtained by analytical continuation from the result valid for $m^2 > eB$, i.e. by substituting $|m^2 - eB| \mapsto (m^2 - eB - i\varepsilon)$. Also notice that the dependence on the cut-off Λ is independent of the magnitude of $m^2 - eB$, so the renormalization will not be affected. The imaginary part in the effective Lagrangian is telling us that the configuration of a spin 1, gyro-magnetic ratio 2 particle in a strong ($eB > m^2$) static uniform magnetic field is unstable. This problem has been addressed earlier in the literature, see Section 6 for references and a discussion. The conclusion is that a condensate will be formed in the unstable mode. The magnetization of this condensate will alter the background magnetic field so that it is no longer uniform on a microscopic scale. The energy of the lowest mode will then become positive so that the instability is removed. However, the net change in the magnetic field is found to be small, so we simply neglect this here. We are thus assuming our results to be valid also in vector QED, and neglect the appearance of the imaginary part. The coefficient $\hat{\chi}$ in front of $(eB)^2\ln(\Lambda^2/m^2)$ obtained from Eq. (4.39) agrees with Eq. (4.26).

4.2 The Mass-less Case

Let us now consider the mass-less limit of spinor, scalar and vector QED. For vector bosons it is then necessary to choose the gauge $D^\mu W_\mu = 0$, in order for the wave-functions to be of the same form as in Eq. (4.7). This will actually introduce Faddeev–Popov ghosts. In for example Ref. [17] it was shown that the contribution from these ghosts exactly cancel the contribution from the unphysical degrees of freedom. This means that if we restrict the sum over polarizations to $s = \pm 1$, we do not need to consider ghosts.

In this mass-less case we may not subtract the next term in the expansion of Eq. (4.13), since this would cause a divergence for large t. Instead we substitute $x = eBt$ in Eq. (4.12) and integrate by parts to find

$$\mathcal{L}_{\rm vac,b} = \frac{1}{2}\hat{\chi}(eB)^2 \ln\left(\frac{\Lambda^2}{eB}\right) + \frac{C'}{2}(eB)^2 \quad , \tag{4.43}$$

where the u.v. finite constant is

$$C' \equiv \frac{(-1)^{2\sigma}}{8\pi^2}\sum_s \int_0^\infty dx \ln x \, \frac{d}{dx}\left[\frac{\exp(\gamma s x)}{x \sinh x} - \frac{1}{x^2}\right] \quad . \tag{4.44}$$

In the mass-less limit, vector QED will be unstable for all values of B. Since only two different polarizations $s = \pm 1$ now are possible, we write

$$\mathcal{L}_{\rm vac}^{\rm vec} = 2\mathcal{L}_{\rm vac}^{\rm scal} + \Delta\mathcal{L}_{\rm vac}^{\rm vec} \quad . \tag{4.45}$$

We may directly take the mass-less limit in Eq. (4.42). The result is that Eq. (4.43) still is valid, but with an imaginary part in the constant C'. This is of no surprise since the renormalization is due to ultra-violet divergences, i.e. $p_z^2 \gg eB$. Again we shall neglect this imaginary part and the instability. The effective Lagrangian of Eq. (4.43) is what was considered in Ref. [2] (but in terms of the vacuum energy). Nielsen [2] extracted from here the leading bare vacuum magnetic susceptibility

$$\chi_{\rm vac,b} \simeq e_b^2 \hat{\chi} \ln \frac{\Lambda^2}{eB} \quad . \tag{4.46}$$

For mass-less vector bosons we find explicitly

$$\hat{\chi}^{\rm vec} = \frac{11}{24\pi^2} \quad . \tag{4.47}$$

This has the opposite sign as compared to spinor and scalar QED. Bare vector QED (that we in Section 6 will relate to QCD) will thus be paramagnetic, that corresponds to anti-screening and asymptotic freedom. However, performing the renormalization we have

$$\mathcal{L}_{\rm vac} = -\frac{1}{2}\hat{\chi}(eB)^2 \ln \frac{eB}{\lambda^2} \quad . \tag{4.48}$$

The sign of the renormalized vacuum magnetic susceptibility, i.e. if we have a paramagnetic ($\chi_{\rm vac} > 0$) or diamagnetic ($\chi_{\rm vac} < 0$) vacuum, will thus again depend on the relative magnitude between the field strength eB, and the renormalization scale λ^2.

Asymptotic freedom really means anti-screening with the effective charge vanishing at vanishing distance (infinitely large momentum scale). From the RGE this is obvious since $e(\lambda) = 0$ is a fixed point. We may also compare the effective charge at two different scales

$$\frac{e^2_{\rm eff}(eB)}{e^2_{\rm eff}(eB_0)} = \left[1 + e^2_{\rm eff}(eB_0)\hat{\chi}\ln\left(\frac{eB}{eB_0}\right)\right]^{-1} \quad . \tag{4.49}$$

If $\hat{\chi} > 0$, as in vector QED and QCD this quotient is vanishing as $eB \to \infty$, i.e. we have asymptotic freedom. In spinor and scalar QED with $\hat{\chi} < 0$, the coupling appears to grow infinitely large at a finite (but extraordinary large) value of the magnetic field. However, perturbation theory (we are only considering one loop effects here) cannot be extrapolated that far.

5 Thermal Spinor, Scalar and Vector QED

We shall in this section consider the contributions to the effective Lagrangian due to finite temperature and density of particles. We may on this one loop level use the naive real time formalism. In Ref. [13] it was shown that the substitution in the fermion vacuum propagator

$$f_F(\omega) = \frac{i}{\omega^2 - E^2 + i\varepsilon} \mapsto -2\pi\delta(\omega^2 - E^2)f_F(\omega) \quad , \tag{5.1}$$

in order to obtain the thermal part of the propagator, works also in a magnetic field. Here the fermion one-particle distribution in thermal equilibrium is the Fermi–Dirac distribution

$$f_F(\omega) = \frac{\Theta(\omega)}{e^{\beta(\omega-\mu)} + 1} + \frac{\Theta(-\omega)}{e^{\beta(-\omega+\mu)} + 1} \quad , \tag{5.2}$$

where μ is the chemical potential, and $1/\beta = T$ is the temperature. In the bosonic case we substitute similarly

$$\frac{i}{\omega^2 - E^2 + i\varepsilon} \mapsto 2\pi\delta(\omega^2 - E^2)f_B(\omega) \quad , \tag{5.3}$$

where f_B, in thermal equilibrium, is the Bose–Einstein distribution

$$f_B(\omega) = \frac{\Theta(\omega)}{e^{\beta(\omega-\mu)} - 1} + \frac{\Theta(-\omega)}{e^{\beta(-\omega+\mu)} - 1} \quad . \tag{5.4}$$

Performing these substitutions in Eq. (4.5), we may use the δ-function to integrate over ω. Integrating with respect to m^2, the constant of integration is determined by the fact that

the exponential suppression from the particle distributions requires $\mathcal{L}_{\rm mat} \to 0$, as $m \to \infty$. We then find the contribution from the heat and charge bath to the effective Lagrangian

$$\mathcal{L}_{\rm mat} = \frac{eB}{(2\pi)^2} \sum_s \sum_{n=0}^{\infty} \int dp_z \frac{p_z^2}{E_{n,s}} \left[\frac{1}{e^{\beta(E_{n,s}-\mu)} - (-1)^{2\sigma}} + \frac{1}{e^{\beta(E_{n,s}+\mu)} - (-1)^{2\sigma}} \right] \quad . \tag{5.5}$$

Integrating by parts with respect to p_z we recognize the free-energy density

$$\begin{aligned}\mathcal{L}_{\rm mat} \equiv \frac{1}{\beta V} \ln Z \quad &= \quad \frac{(-1)^{2\sigma+1}}{\beta} \frac{eB}{(2\pi)^2} \sum_s \sum_{n=0}^{\infty} \int dp_z \left\{ \ln\left[1 - (-1)^{2\sigma} e^{-\beta(E_{n,s}-\mu)}\right] \right. \\ &\quad \left. + \ln\left[1 - (-1)^{2\sigma} e^{-\beta(E_{n,s}+\mu)}\right] \right\} \quad , \end{aligned} \tag{5.6}$$

where Z is the partition function of the gas. Let us now split $\mathcal{L}_{\rm mat}$ into

$$\mathcal{L}_{\rm mat} \equiv \mathcal{L}_{\rm mat,0} + \mathcal{L}_{\rm mat,1} \quad , \tag{5.7}$$

where $\mathcal{L}_{\rm mat,0}$ is the field independent part

$$\mathcal{L}_{\rm mat,0} = \frac{(-1)^{2\sigma+1}}{\beta} \sum_s \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left\{ \ln\left[1 - (-1)^{2\sigma} e^{-\beta(E(p)-\mu)}\right] + \ln\left[1 - (-1)^{2\sigma} e^{-\beta(E(p)+\mu)}\right] \right\} \ , \tag{5.8}$$

whith $E(p) = \sqrt{m^2 + \mathbf{p}^2}$. We are particularly interested in the high temperature expansion of the effective Lagrangian. We shall consider the massive case for spinor, scalar and vector QED in Section 5.1, and investigate the corresponding mass-less limits in Section 5.2. In every case the contribution from thermal photons

$$\mathcal{L}_{\rm mat}^{\rm phot} = \frac{2\pi^2}{45} T^4 \quad , \tag{5.9}$$

should be added to the effective Lagrangian. In Section 5.3 we consider also a degenerate fermion gas at low temperature and high density.

5.1 The Massive case

The case of massive fermions in a magnetic field was treated in Ref. [13]. Here we just quote the final result

$$\begin{aligned}\mathcal{L}_{\rm mat,1}^{\rm ferm} \quad &= \quad \int d\omega \Theta(\omega^2 - m^2) f_F(\omega) \left\{ \frac{1}{4\pi^{5/2}} \int_0^\infty \frac{dt}{t^{5/2}} e^{-t(\omega^2-m^2)} [eBt \coth(eBt) - 1] \right. \\ &\quad \left. - \frac{1}{2\pi^3} \sum_{l=1}^{\infty} \left(\frac{eB}{l}\right)^{3/2} \sin\left(\frac{\pi}{4} - \pi l \frac{\omega^2 - m^2}{eB}\right) \right\} \quad . \end{aligned} \tag{5.10}$$

The high temperature, weak field limit $\{T^2 \gg m^2 \gg eB;\ \mu = 0\}$ was considered in Ref. [1], with the result

$$\mathcal{L}_{\rm mat}^{\rm ferm} + \tilde{\mathcal{L}}_{\rm vac}^{\rm ferm} = \frac{7\pi^2}{180} T^4 - \frac{1}{2} \hat{\chi}^{\rm ferm} (eB)^2 \ln\left(\frac{T^2}{m^2}\right) + \mathcal{O}[(eB)^2] \quad . \tag{5.11}$$

Notice here that in the high-temperature effective Lagrangian, terms with higher powers of eB are suppressed as $(eB)^2(eB/T^2)^{n-2}, \quad n \geq 4$. To leading order there are thus no non-linear electro-magnetic interactions in high temperature QED, in agreement with a diagrammatic analysis [23]. Also notice that the $\ln m^2$ terms will cancel when this is added to the effective Lagrangian in Eq. (4.28), and that the effective charge obtained from this will satisfy the lowest order RGE with the scale $\lambda = T$.

Let us now consider the case of massive scalar QED. For the sake of completeness we here quote the result (see e.g. Ref. [13])

$$\begin{aligned}\mathcal{L}^{\rm scal}_{\rm mat,1} &= \int d\omega \Theta(\omega^2 - m^2 - eB) f_B(\omega) \left\{ \frac{1}{8\pi^{5/2}} \int_0^\infty \frac{dt}{t^{5/2}} e^{-t(\omega^2 - m^2)} \left[\frac{eBt}{\sinh(eBt)} - 1 \right] \right. \\ & \left. - \frac{1}{4\pi^3} \sum_{l=1}^{\infty} \left(\frac{eB}{l} \right)^{3/2} \sin\left(\frac{\pi}{4} - \pi l \frac{\omega^2 - m^2 - eB}{eB} \right) \right\} \quad . \end{aligned} \tag{5.12}$$

In the high temperature, weak field limit, this gives [1]

$$\begin{aligned}\mathcal{L}^{\rm scal}_{\rm mat} + \tilde{\mathcal{L}}^{\rm scal}_{\rm vac} &= \frac{\pi^2}{45} T^4 - \frac{1}{2} \hat{\chi} (eB)^2 \left\{ \ln \left(\frac{4\pi T}{m} \right)^2 - \gamma_E + \mathcal{O}\left[\left(\frac{m}{T} \right)^2 \right] \right\} \\ & - \frac{T}{m} \frac{(eB)^2}{48\pi} \left\{ 1 + \mathcal{O}\left[\left(\frac{eB}{m^2} \right)^2 \right] + \mathcal{O}\left[\frac{m}{T} \left(\frac{eB}{T^2} \right)^2 \right] \right\} \quad . \end{aligned} \tag{5.13}$$

In the massive vector case we again write $\mathcal{L}^{\rm vec}_{\rm mat} = 3\mathcal{L}^{\rm scal}_{\rm mat} + \Delta\mathcal{L}^{\rm vec}_{\rm mat}$. For $m^2 > eB$ and $\mu = 0$, we have

$$\Delta\mathcal{L}^{\rm vec}_{\rm mat} = \frac{eB}{\pi^2} \int_0^\infty dp_z p_z^2 \left[\frac{1}{E_{0,-1}} \frac{1}{e^{\beta E_{0,-1}} - 1} - \frac{1}{E_{0,0}} \frac{1}{e^{\beta E_{0,0}} - 1} \right] \quad . \tag{5.14}$$

The calculations performed in order to find the high temperature expansion of this expression are explicitly shown in Appendix A.1. The final result reads

$$\begin{aligned}\Delta\mathcal{L}^{\rm vec}_{\rm mat} &= -\frac{(eB)^2}{4\pi^2} \ln \left[\frac{T^2 (4\pi)^2}{\sqrt{|m^4 - (eB)^2|}} \right] + \frac{eBT}{2\pi} (\sqrt{m^2 + eB} - \sqrt{|m^2 - eB|}) \\ & + \frac{eBm^2}{8\pi^2} \ln \left(\frac{m^2 + eB}{|m^2 - eB|} \right) - \frac{(eB)^2}{2\pi^2} \left(\frac{1}{2} - \gamma_E \right) + eBT^2 \mathcal{O}\left(\frac{m^2 + eB}{T^2} \right)^{3/2} \\ & + i\Theta(eB - m^2) \left[\frac{eBT\sqrt{eB - m^2}}{2\pi} - \frac{eB(eB - m^2)}{8\pi} \right] \quad . \end{aligned} \tag{5.15}$$

Notice that the $\ln(m^2 \pm eB)$ terms cancel between Eq. (4.42) and Eq. (5.15). For $eB > m^2$ the lowest energy mode will become unstable, resulting in the imaginary part as depicted above. The result follows also in this case by analytical continuation $m^2 \to m^2 - i\varepsilon$ in expression valid for $eB < m^2$. Notice that the T-independent imaginary part exactly cancels the imaginary part of the corresponding vacuum contribution in Eq. (4.42), and that the resulting imaginary part is increasing linearly with the temperature.

Of course such an imaginary part is not acceptable. Since a full treatment of this imaginary part is out of the scope here we are left with two alternatives. Either neglecting the imaginary part, or neglecting the total contribution from the tachyonic mode, that also has a real part. In the former approach we may immediately use Eq. (5.15). The latter approach is considered in the mass-less case in the following Section 5.2. The massive case (that is relevant for spontaneously broken gauge theories) can be treated in full analogy.

5.2 The Mass-less case

The explicit calculations performed in order to obtain the high temperature expansion in mass-less spinor and scalar QED are presented in Appendix A.2. Adding the thermal contribution for $B = 0$, $\mathcal{L}_{\text{mat},0}$, The final result reads in the spinor case

$$\mathcal{L}^{\text{ferm}}_{\text{vac}} + \mathcal{L}^{\text{ferm}}_{\text{mat}} = \frac{7\pi^2}{180}T^4 - \frac{1}{2}\hat{\chi}^{\text{ferm}}(eB)^2 \ln\frac{T^2}{\lambda^2} + \mathcal{O}[(eB)^2] \quad . \tag{5.16}$$

Just like in the massive case, all non-linear electro-magnetic interactions from the vacuum part has been cancelled by contributions from the thermal part. In the scalar case we find

$$\mathcal{L}^{\text{scal}}_{\text{vac}} + \mathcal{L}^{\text{scal}}_{\text{mat}} = \frac{\pi^2}{45}T^4 - \frac{\sqrt{2}-1}{2\pi}|\zeta(-\frac{1}{2})|(eB)^{3/2}T - \frac{1}{2}\hat{\chi}^{\text{scal}}(eB)^2 \ln\frac{T^2}{\lambda^2} + \mathcal{O}[(eB)^2] \quad . \tag{5.17}$$

Again the $\ln[(eB)^2/\lambda^2]$ has been cancelled by thermal contributions, but there is in this case also the term linear in T.

We are now left with the vector bosons. The configuration of mass-less vector bosons in a static uniform magnetic field is unstable for all field-strengths. We will here neglect the contribution for $p_z^2 < eB - m^2$ in the lowest Landau level in order to avoid the imaginary part. However, this may affect the high temperature behavior, that is not solely governed by large momenta, but also by small energies when the Bose-Einstein distribution is becoming very large. The final result of the calculations performed in Appendix A.3 reads

$$\Delta\mathcal{L}^{\text{vec}}_{\text{mat}} = -\frac{(eB)^2}{\pi^2}\left\{\frac{T}{\sqrt{eB}}\left[\frac{1}{2}\ln\frac{T^2}{4eB} + 1 - \frac{\pi}{2}\right] + \frac{1}{4}\ln\frac{T^2}{eB} + \mathcal{O}(1)\right\} \quad . \tag{5.18}$$

The same expression is obtained if we substitute $m^2 \to -i\varepsilon$ in Eq. (5.15), and subtract the corresponding integral $\int_0^{\sqrt{eB}}$ analytically continued to imaginary energy.

5.3 The Dense Degenerate fermion Gas

Chodos et al. [24] has suggested that the fermion matter contribution in the limit of large chemical potential may be of importance in heavy-ion collisions. Let for simplicity $T = 0$. The leading behavior of the first term in Eq. (5.10) is then [13]

$$\mathcal{L}_{\text{reg}} \simeq -\frac{1}{2}\hat{\chi}^{\text{ferm}}(eB)^2 \ln\frac{\mu^2}{m^2} \quad . \tag{5.19}$$

We may write the oscillating second term in Eq. (5.10) as

$$\mathcal{L}_{\rm osc} = -\frac{(eB)^{3/2}}{2\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}\,{\rm Im}\left\{\exp\left[i\left(\frac{\pi}{4}-\pi n\frac{m^2}{eB}\right)\right]J_n\right\} \quad . \tag{5.20}$$

Here we have defined

$$\begin{aligned} J_n &\equiv \int_m^\mu d\omega \exp\left[-i\pi n\frac{\omega^2}{eB}\right] \\ &= e^{-i\pi/4}\sqrt{\frac{eB}{2n}}\left\{{\rm erf}\left[e^{i\pi/4}\sqrt{\frac{n\pi\mu^2}{eB}}\right]-{\rm erf}\left[e^{i\pi/4}\sqrt{\frac{n\pi m^2}{eB}}\right]\right\} \quad , \end{aligned} \tag{5.21}$$

where error functions were identified. Let us now neglect the second term that is independent of the chemical potential. Using the asymptotic expansion of the error function [43] for $\pi\mu^2 \gg eB$ we find the leading term at large chemical potentials

$$\mathcal{L}_{\rm osc} \simeq -\frac{(eB)^{5/2}}{4\pi^4\mu}\sum_{n=1}^{\infty}\frac{1}{n^{5/2}}\sin\left(\frac{\pi}{4}+n\pi\frac{\mu^2-m^2}{eB}\right) \quad . \tag{5.22}$$

This is suppressed at large μ, but performing the derivative with respect to eB we find the leading behavior, with ${\rm mod}[A] \equiv A - {\rm int}[A]$,

$$\begin{aligned} \frac{M_{\rm osc}}{e^2B} &\simeq \frac{\mu}{\sqrt{eB}}\frac{1}{4\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}\sin\left(\frac{\pi}{4}-2n\pi\frac{\mu^2-m^2}{2eB}\right) & (5.23)\\ &= -\frac{\mu}{\sqrt{2eB}}\frac{1}{3\pi^2}\zeta\left(-\frac{1}{2},{\rm mod}\left[\frac{\mu^2-m^2}{2eB}\right]\right). & (5.24) \end{aligned}$$

In Ref. [24] the results of Ref. [13] was used to find Eq. (5.23), using another method. Chodos et al. [24] suggest that the effective charge obtained from this, increasing linearly with μ for $\mu^2 = m^2 + 2eBn,\ n \in \mathbf{Z}$, could be of relevance in heavy ion collisions. In Ref. [24] it was claimed that this is valid only for $\mu^2 \gg eB \gg m^2$, but here we have shown that this is the leading behavior for $\mu^2 \gg eB,\ m^2$ irrespective of the relative magnitude of eB and m^2. The effective charge then seems to become divergent as $eB \to 0$. This must be an artifact due to one out of two possibilities.

1. The break-down of perturbation theory, as the coupling is becoming stronger.

2. The derivative in the definition of the effective charge. If we instead would define the effective charge by $1/e_{\rm eff}^2 = -2\mathcal{L}_{\rm eff}/(eB)^2$, then the contribution from the oscillating part is suppressed at large chemical potentials.

Furthermore, the linearly increasing amplitude in $M_{\rm osc}$ is a result of the sharp de Haas–van Alphen oscillations at the Fermi surface, that will be smoothed out at finite temperature. The ζ-function in Eq. (5.23) takes its minimal value $\zeta(-1/2,0)=\zeta(-1/2,1)\simeq -0.208$, and its maximal value $\zeta(-1/2,0.3027)\simeq 0.0934$. The effective charge is thus oscillating, and we doubt its physical significance in this limit.

6 QCD with a Background Magnetic Field

Quantum Chromodynamics with a background magnetic field has been extensively studied in the literature. It is well-known that the mean-field equation (3.8) has a non-vanishing solution for B, even in the absence of an external field (i.e. $H = 0$) in QCD. The vacuum energy is lower in this state than for $B = 0$, as pointed out in Ref. [25]. However a tachyonic mode appears , cf. Eq. (4.4), that causes an imaginary part in the effective potential [3], signaling the instability of this configuration. This unstable mode has been suggested to be removed by a (1+1) dimensional dynamical Higgs mechanism [26]. The corresponding condensate shows a domain-like structure, and may form a quantum liquid of magnetic flux tubes [27, 28]. A corresponding treatment has also been done in the electro-weak theory (e.g. Refs. [29, 30]). Here the change in the magnetic field due to the condensate, that assumes a lattice structure, is found to be small compared to the uniform field. It has also been suggested in another approach that the tachyonic mode could be stabilized by radiative corrections [31, 32], or by the condensation of an auxiliary field [33]. In Refs. [34, 35] the imaginary part was found to appear also for a non-Abelian like background field. However, it was argued [34] that the imaginary part originates in the abuse of the formula $\mathcal{L}_{\text{eff}} \propto \ln \det G^{-1}$, where G is a propagator, valid only for positive definite G. It was concluded [34] that the contribution from the unstable modes and the imaginary part should not be trusted. In Ref. [19] the imaginary part was shown to vanish for large enough values of a color condensate A_0.

Since a full treatment of the tachyonic mode is out of the scope of this monograph, some approximations or assumptions are required. In for example Refs. [14, 15] the contribution from the tachyonic mode was neglected, as we have done here in Section 5.2. However, in Refs. [16, 17] it was argued that the contribution of the unstable mode should be taken into account by analytical continuation, as in Eq. (5.15). Starting from the partition function or the free energy of the quark gluon plasma, cf. Eq. (5.6), it seems very unnatural to include an *unphysical* tachyonic mode. Due to the presence of the thermal distribution function the thermal contribution of the tachyonic mode will not be purely imaginary, unlike the vacuum contribution. In Ref. [36] the existence of a zero-energy mode, after the removal of the tachyonic mode, was pointed out. This suggests that including contributions only from $p_z^2 + m^2 - eB \geq 0$ in the lowest energy mode is a reasonable approximation. However, this will exclude the contribution from the plausible condensate. We shall therefore briefly comment also on the corresponding results when the real part of the contribution from the unstable mode is included. These problems may be solved by introducing a thermal gluon mass, large enough to remove the instability. The effects of such a mass are considered in Section 6.4.

Usually a temperature dependent (as well as RGE running) coupling is obtained from the vertex correction. In the background field formalism here employed the vertex cor-

rection is related to the vacuum polarization. We shall therefore consider the vacuum polarization, and review the obtained effective couplings in Section 6.3.

The conserved charge in QCD may correspond to the color charge, the baryon number or the quark flavor. Each quark is carrying the color charge “1” of some color, and the baryonic number 1/3 that relates the baryon chemical potential to the color chemical potential. Nature only seems to allow for equal amounts of the different colors, that corresponds to equal chemical potentials. However, the gluons carry equal amount of color and anti-color (but not the respective), resulting in a vanishing chemical potential for them, as well as their linear combination in terms of the W fields. The W bosons carry an equal amount of color and anti-color, in the ideal combination for interactions with the external field. The problem with Bose–Einstein condensation in the lowest energy mode thus never occurs, since $\mu = 0$ for the gluons and W bosons. Nevertheless, we shall here mainly focus on the high temperature situation with vanishing charge density (i.e. $\mu = 0$). Since a finite chemical potential only affects the quarks, we may in this case immediately use the results obtained for QED in Section 5.3.

6.1 Connections with Spinor and Vector QED

We shall here first relate the theory of mass-less spin $\sigma = 1$, gyro-magnetic ratio $\gamma = 2$ bosons interacting with a background magnetic field, to $SU(N_c)$ Yang–Mills theory in a background chromo-magnetic field (cf. Refs. [2, 37]). The Lagrangian of $SU(N_c)$ Yang–Mills, is

$$\mathcal{L}_{\mathrm{YM}}^{N_c} = -\frac{1}{4}\sum_{a=1}^{N_c^2-1} G^a_{\mu\nu}G^{a\mu\nu} \equiv \sum_{a=1}^{N_c^2-1}\mathcal{L}^{(a)} \quad , \tag{6.1}$$

where

$$G^a_{\mu\nu} = \partial_\mu E^a_\nu - \partial_\nu E^a_\mu + g\sum_{b,c} f^{abc}E^b_\mu E^c_\nu \quad . \tag{6.2}$$

We shall now consider this theory in the presence of a background chromo-magnetic field, that we choose in the $a' \equiv N_c^2 - 1$ direction in color space, i.e.

$$E_\mu^{N_c^2-1} \equiv A_\mu + \tilde{E}_\mu^{N_c^2-1} \quad . \tag{6.3}$$

Here A_μ is the vector potential corresponding to a static uniform magnetic field, and $\tilde{E}_\mu^{N_c^2-1}$ is the quantum field. As in the Abelian theories considered before, we are interested in the one-loop effective Lagrangian for this background magnetic field A_μ. To the one-loop order we may neglect every occurrence of g, when not in the combination gA_μ. We then find

$$\mathcal{L}^{(a')} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}gF^{\mu\nu}\sum_{j=1}^{N_c^2-2}\sum_{k=1}^{N_c^2-2} f^{a'jk}E^j_\mu E^k_\nu \quad , \tag{6.4}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the (color) electro-magnetic field-strength tensor. As in the Abelian theories considered above we have not explicitly written out the radiation field $\tilde{E}_\mu^{N_c^2-1}$, but we must remember to include its thermal contribution. In the limit $g = 0$ the Lagrangian of Eq. (6.1) corresponds to $(N_c^2 - 1)$ "photons", that gives the thermal contribution for $B = 0$ to the effective Lagrangian. For $j' \neq a'$ we find

$$\mathcal{L}^{(j')} = -\frac{1}{4}\left[\partial_\mu E_\nu^{j'} - \partial_\nu E_\mu^{j'} - g\sum_{k=1}^{N_c^2-1} f^{a'j'k}(A_\mu E_\nu^k - A_\nu E_\mu^k)\right]^2 \quad , \tag{6.5}$$

where we have used the total anti-symmetry of the structure constants $f^{j'a'k} = -f^{a'j'k}$. Generally we may choose the generators of $SU(N_c)$, such that $f^{a'j'k}$ is non-vanishing only for one $k = k'$, for fixed a' and j'. Let us now define

$$\begin{aligned} E_\mu^{j'} &\equiv \frac{1}{\sqrt{2}}[W_\mu^{(j',k')} + W_\mu^{(j',k')\dagger}] \quad , \\ E_\mu^{k'} &\equiv \frac{1}{i\sqrt{2}}[W_\mu^{(j',k')} - W_\mu^{(j',k')\dagger}] \quad . \end{aligned} \tag{6.6}$$

In terms of the charged vector boson field W we have

$$\mathcal{L}^{(j')} + \mathcal{L}^{(k')} = -\frac{1}{2}[D_\mu^* W_\nu^{(j',k')\dagger} - D_\nu^* W_\mu^{(j',k')\dagger}][D^\mu W^{(j',k')\nu} - D^\nu W^{(j',k')\mu}] \quad , \tag{6.7}$$

where we have defined the covariant derivative

$$D_\mu W_\nu^{(j',k')} \equiv (\partial_\mu - igf^{a'j'k'}A_\mu)W_\nu^{(j',k')} \quad . \tag{6.8}$$

Adding the relevant term from $\mathcal{L}^{(a')}$ in Eq. (6.4), we find

$$\begin{aligned} \mathcal{L}^{(j',k')} - \frac{1}{4}F^2 &= -\frac{1}{4}F^2 - \frac{1}{2}[D_\mu^* W_\nu^{(j',k')\dagger} - D_\nu^* W_\mu^{(j',k')\dagger}][D^\mu W^{(j',k')\nu} - D^\nu W^{(j',k')\mu}] \\ &\quad -igf^{a'j'k'}F^{\mu\nu}\frac{W_\mu^{(j',k')\dagger}W_\nu^{(j',k')} - W_\mu^{(j',k')}W_\nu^{(j',k')\dagger}}{2} \quad . \end{aligned} \tag{6.9}$$

Comparing with Eq. (4.3), we see that this is exactly the Lagrangian for a spin $\sigma = 1$, gyro-magnetic ratio $\gamma = 2$ boson of charge $-gf^{a'j'k'}$, interacting with a background field, described by A_μ. The terms in the original Lagrangian $\mathcal{L}_{\rm YM}^{N_c}$ describing the self-interaction of the E^a fields, that we here have discarded when considering the one-loop effective Lagrangian for A_ν, are essential for the renormalizability of the theory [37]. The total Lagrangian $\mathcal{L}_{\rm YM}^{N_c}$ is then obtained by summing over all such pairs (j', k')

$$\mathcal{L}_{\rm YM}^{N_c} = -\frac{1}{4}F^2 + \sum_{(j,k)} \mathcal{L}^{(j,k)} \quad , \tag{6.10}$$

where (j,k) are the pairs with non-vanishing $f^{a'jk}$, for $a' = N_c^2 - 1$. Again using the total anti-symmetry of the structure constants, the total charge squared of these particles is

$$e_{\text{vec}}^2 = g^2 \sum_{(j,k)} (f^{a'jk})^2 = \frac{g^2}{2} \sum_{j=1}^{N_c^2-1} \sum_{k=1}^{N_c^2-1} f^{a'jk} f^{a'jk} \quad . \tag{6.11}$$

Now the $SU(N_c)$ generators in the adjoint representation are $(T^a_{\text{vec}})_{jk} = f^{ajk}$. We thus find a group invariant squared charge, independent of the direction a' in $SU(N_c)$

$$e_{\text{vec}}^2 \delta^{ab} = \frac{g^2}{2} \text{Tr}\,[T^a_{\text{vec}} T^b_{\text{vec}}] = \frac{g^2}{2} N_c \delta^{ab} \quad . \tag{6.12}$$

We now wish to couple fermions in the fundamental representation to our Yang–Mills theory. The corresponding Lagrangian reads

$$\mathcal{L}_{\text{YM}}^{\text{ferm}} = \overline{\psi}(i\partial\!\!\!/ + g \sum_{a=1}^{N_c^2-1} E\!\!\!/^a T^a_f - m)\psi \quad , \tag{6.13}$$

where the color indices $j = 1, 2, \ldots, N_c$ have been suppressed. In the background field A_μ of Eq. (6.3), we again neglect g when not combined in gA_μ, and find

$$\mathcal{L}_{\text{YM}}^{\text{ferm}} = \overline{\psi}(i\partial\!\!\!/ + gA\!\!\!/ T^{a'}_{\text{ferm}} - m)\psi \quad . \tag{6.14}$$

Generally we may choose $T^{a'}_{\text{ferm}}$ diagonal, i.e. $T^{a'}_{\text{ferm}} = \text{diag}(t_1, t_2, \ldots, t_{N_c})$. Comparing with Eq. (4.1), we see that the Lagrangian in Eq. (6.14) corresponds to N_c fermions with charges $e = -g(t_1, t_2, \ldots, t_{N_c})$, coupled to the external field described by A_μ. The squared sum of their charges is then also group invariant

$$e_{\text{ferm}}^2 \delta^{ab} = \left(g^2 \sum_{j=1}^{N_c} t_j^2 \right) \delta^{ab} = g^2 \text{Tr}\,[T^a_{\text{ferm}} T^b_{\text{ferm}}] = \frac{g^2}{2} \delta^{ab} \quad . \tag{6.15}$$

Invariance under background field gauge transformations requires the product eB to be invariant under renormalization (see e.g. Ref. [9]). We may then immediately use the results of the previous Sections. We shall for simplicity use the relations between quadratic charges in Eqs. (6.11, 6.15). When terms not quadratic in the coupling appear in the effective Lagrangian, the correct approach is to sum over the moduli of the constituent charges to the power considered, as we assume $eB > 0$. However, when not only containing even powers of the coupling e, the result is not group-invariant. This means that it depends on the specific direction in color space chosen for the magnetic field. Any such result is dubious, and probably unphysical.

6.2 The Effective Coupling in Thermal QCD

In the previous Section we related QCD with a background (chromo-) magnetic field to the corresponding cases in spinor and vector QED. However, for the contributions to the effective Lagrangian independent of the magnetic field, we must instead identify and sum over all degrees of freedom. We shall here consider N_f flavors of quarks with possibly different masses m_f. They all come in N_c different colors. The number of gluons is N_c^2-1. Since we only are summing over the true degrees of freedom, either in the vacuum energy, or in the free energy, no Faddeev–Popov ghosts are needed. The effective Lagrangian for $SU(N_c)$ Yang-Mills with a background color magnetic field, and N_f flavors of quarks in the fundamental representation may thus be written as

$$\mathcal{L}_{\text{eff}}^{\text{QCD}} = \mathcal{L}_{\text{tree}}+N_c\sum_f \mathcal{L}_{\text{mat},0}^{\text{ferm}}+(N_c^2-1)\mathcal{L}_{\text{mat},0}^{\text{vec}}+\sum_f \mathcal{L}_{\text{mat},1}^{\text{ferm}}(e_{\text{ferm}}^2=\frac{g^2}{2})+\mathcal{L}_{\text{mat},1}^{\text{vec}}(e_{\text{vec}}^2=\frac{N_c}{2}g^2) \quad , \tag{6.16}$$

with $m\mapsto m_f$ in the different terms in the sum over flavors. In vacuum the corresponding renormalization gives the correct lowest order QCD β-function

$$\lambda\frac{dg(\lambda)}{d\lambda}\equiv\beta^{\text{QCD}}[g(\lambda)]=-\hat{\chi}^{\text{QCD}}g^3(\lambda) \quad , \tag{6.17}$$

where we obtain from Eqs. (4.24, 4.47) and Eqs. (6.12, 6.15)

$$\hat{\chi}^{\text{QCD}}=\frac{N_c}{2}\hat{\chi}^{\text{vec}}+\frac{N_f}{2}\hat{\chi}^{\text{ferm}}=\frac{11N_c-2N_f}{48\pi^2} \quad . \tag{6.18}$$

In the high temperature limit $\{T^2\gg eB,\ m_f^2\}$ for all flavors of quarks, we find when neglecting the contribution from the tachyonic mode

$$\begin{aligned}\mathcal{L}_{\text{eff}}^{\text{QCD}} &= \pi^2T^4\left\{\frac{7}{180}N_cN_f+\frac{2}{45}(N_c^2-1)\right\}-\frac{1}{2}\frac{(gB)^2}{g^2(\lambda)}-\frac{1}{2}\hat{\chi}^{\text{QCD}}(gB)^2\ln\frac{T^2}{\lambda^2}\\ &-\frac{1}{2\pi^2}\left(\frac{N_c}{2}\right)^{3/4}(gB)^{3/2}T\left[\ln\frac{T^2}{4gB}+2\pi(\sqrt{2}-1)|\zeta(-1/2)|+2-\pi\right]+\mathcal{O}[(gB)^2].\end{aligned} \tag{6.19}$$

Here the ordinary renormalization at $T=0,\mu=0$ has been performed at the renormalization scale λ. The dependence of λ in $g(\lambda)$ is cancelling the explicit λ dependence, so that the effective Lagrangian is independent of λ, as follows from its definition. Notice that the $(gB)^2\ln(m^2)$ terms cancel between Eq. (4.28) and Eq. (5.11), so that there is no dependence of the different quark masses whatsoever, as long as $T^2\gg m_f^2$, and that the same result follows from the mass-less case in Eq. (5.16). Due to the appearance of $(gB)^{3/2}$ it is not quite correct to use the squared average charge for the gluons. In $SU(3)$ with B in the $N_c^2-1=8$ direction in color space, we have the relevant structure constants $f^{458}=\sqrt{3}/2=f^{678}$ (see e.g. Ref. [37]). We should thus substitute

$(N_c/2)^{3/4} \equiv (3/2)^{3/4} \mapsto 2\,(3/4)^{3/4}$ in the term containing $(gB)^{3/2}T$, but this depends on the direction in color space. In Refs. [14, 15], where the tachyonic mode was neglected, no high temperature expansion was made, and the graphs presented do not extend to temperatures high enough, that we could compare our result with theirs.

Let us now use Eq. (3.6) to define a field and temperature dependent effective coupling $g_{\rm eff}(T, eB)$. Comparing the effective charge at different temperature and field strengths, we find

$$\frac{1}{g^2_{\rm eff}(T, eB)} \simeq \frac{1}{g^2_{\rm eff}(T_0, eB_0)} + \hat{\chi}^{\rm QCD} \ln \frac{T^2}{T_0^2} + F(T/eB) - F(T/eB_0) \quad , \tag{6.20}$$

where we have defined

$$F(x) \equiv x \left(\frac{N_c}{2}\right)^{3/4} \frac{3}{4\pi^2} \left[\ln \frac{x^2}{4} - 2\pi(\sqrt{2}-1)|\zeta(-\frac{1}{2})| + 1 - \pi \right] \quad . \tag{6.21}$$

This effective coupling is thus *decreasing* as a function of the temperature. The leading behavior, with $x \equiv T/\sqrt{eB}$ is

$$g^2_{\rm eff}(x) \simeq \frac{g^2_{\rm eff}(x_0)}{1 + g^2_{\rm eff}(x_0) 3 (N_c/2)^{3/4} x \ln(x)/(2\pi^2)} \quad . \tag{6.22}$$

This is a faster decreasing coupling than predicted by the vacuum RGE with the identification $\lambda \mapsto T$. The dominant $T/\sqrt{eB}\, \ln(T^2/eB)$ term is neither present when considering the running coupling obtained from the vacuum polarization (see Section 6.3), nor when including the real part of the contribution from the tachyonic mode as considered next. In this latter case we find instead

$$\mathcal{L}^{\rm QCD}_{\rm mat,1} + \mathcal{L}^{\rm QCD}_{\rm vac} = \frac{(gB)^{3/2}T}{2\pi} 2 \left(\frac{3}{4}\right)^{3/4} [(1+i) + 2(\sqrt{2}-1)\zeta(-1/2)] - \frac{1}{2}\hat{\chi}^{\rm QCD}(gB)^2 \ln \frac{T^2}{\lambda^2} \quad . \tag{6.23}$$

We have here summed over two vector bosons with the charge $e = g\sqrt{3}/2$, to obtain the coefficient in front of $(gB)^{3/2}T$ for comparisons with earlier results. The factor $2(3/4)^{3/4}$ should not be present in $SU(2)$ Yang–Mills theory (with $f^{123} = 1$).

Evaluating numerically we find $[1 + 2(\sqrt{2}-1)\zeta(-1/2)] \simeq 0.827781$, that exactly equals the corresponding numerical coefficient in Ref. [17] (where $SU(2)$ was considered). As a matter of fact, using the same identification with a generalized Riemann ζ function as used here, the coefficient of Ref. [17] may be written as $i + [2^{3/2}|\zeta(-1/2, 3/2)| - 1] \simeq i + 0.827781$. However, we do not agree with the high temperature limit in Ref. [16], and particularly not with the sign of the imaginary part. That sign of the imaginary part is obtained if we substitute $m^2 - eB \to i\varepsilon - eB$ (i.e. opposite to the Feynman prescription in the propagator) of Eq. (5.15). It was pointed out already in Ref. [17] that the different sign on the imaginary part is unphysical, and corresponds to a blow up instead of a decay of the corresponding

configuration. In Ref. [19] (where $SU(2)$ was considered) a coefficient of $(eB)^{3/2}T/2\pi$ in perfect agreement with Eq. (6.23) is obtained (in the limit $A_0 = 0$, that is what here is considered), but the real contribution from $\Delta\mathcal{L}_{\rm mat}^{\rm vec}$ (i.e. "1" in $1+i$) is unfortunately lost in the final result. Probably a typographic error also has caused a coefficient in front of the $(gB)^2\ln(T^2/\lambda^2)$ not in accordance to the RGE in Ref. [19]. Considering the real part, Eq. (6.23) gives an effective charge that is *increasing* with the temperature. The leading behavior of F in Eq. (6.20) is in this case

$$F(x) = x\,2\left(\frac{3}{4}\right)^{3/4}\frac{3}{4\pi}[1+2(\sqrt{2}-1)\zeta(-1/2)] \quad . \tag{6.24}$$

Notice that the effective coupling in Eq. (6.22) and the one obtained from Eq. (6.24) are not invariant under transformations in color space. In Eq. (6.24) we have explicitly stated the result when the magnetic field is chosen in the "8" direction in $SU(3)$ in order to compare with previous results. We feel dubious to the physical significance of such an effective charge.

6.3 The Vacuum Polarization

At finite temperature, the broken Lorentz invariance results in two different possible tensor structures in the vacuum polarization

$$\Pi^{\mu\nu}(k) \equiv P_T^{\mu\nu}\Pi^T(k) + P_L^{\mu\nu}\Pi^L(k) \qquad , \tag{6.25}$$

where $P_T^{\mu\nu}$ and $P_L^{\mu\nu}$ are the spatially transverse and longitudinal polarization operators, respectively. Considering the effective coupling obtained from the vacuum polarization for a gluon with momentum scale $|\mathbf{k}| = \kappa$ in mass-less thermal QCD, we may write

$$\frac{1}{g^2(\kappa,T)} = \frac{1}{g^2(\kappa_0,T_0)} + \hat{\chi}^{\rm QCD}\ln\frac{\kappa^2}{\kappa_0^2} + \tilde{\Pi}(T/\kappa) - \tilde{\Pi}(T_0/\kappa_0) \quad . \tag{6.26}$$

We have here already used the well known form of the vacuum polarization in the absence of matter, leading to Eq. (4.27). Chaichian and Hayashi [9] suggest to use either $g^2\tilde{\Pi} = \Pi_{\rm mat}^L/\kappa^2 = (k^2/\kappa^2)\Pi_{\rm mat}^{00}/\kappa^2$, or $g^2\tilde{\Pi} = \Pi_{\rm mat}^T/\kappa^2 = (\sum_j \Pi_{\rm mat}^{jj} - (k_0^2/\kappa^2)\Pi_{\rm mat}^{00})/(2\kappa^2)$. We shall here only consider the static momentum configuration, $k^0 = 0,\ |\mathbf{k}| = \kappa$.

With the formalism used in this work we may easily calculate $\Pi_{\rm mat}^{00}$ in the limit $k_\nu = 0$. Using the similarity between the way the chemical potential μ and the vector potential A_0 enters the Lagrangian, we find

$$\Pi_{\rm mat}^{00}(k_\nu = 0) = e^2\frac{\partial^2\mathcal{L}_{\rm mat}}{\partial\mu^2} \quad . \tag{6.27}$$

Since we have found the quark masses to be irrelevant in the high temperature limit, let us for simplicity consider the mass-less case only. Performing the derivative in Eq. (A.9), we may then let $\mu = 0$. The field independent part is then obtained as

$$\Pi^{00}_{\mathrm{mat},0} = \frac{e^2}{2\pi^2} T^2 \sum_s \sum_{l=1}^{\infty} \frac{(-1)^{2\sigma(l-1)}}{l^2} \int_0^\infty \frac{dt}{t^2} \exp[-\frac{1}{2t}] \quad . \tag{6.28}$$

This gives $\Pi^{00,\mathrm{scal}}_{\mathrm{mat},0} = \Pi^{00,\mathrm{ferm}}_{\mathrm{mat},0} = e^2T^2/3$, and in the vector case $\Pi^{00,\mathrm{vec}}_{\mathrm{mat},0} = 2e^2T^2/3$. What amounts to the field dependent part we proceed much in the same way as in Appendix A.2. Substituting $u = \beta^2 l^2 eBt/2$, we may directly perform the Poisson resummation, since the $l = 0$ term is vanishing in this case. Again it is necessary to separate out the $k = 0$ term for bosons. In the scalar case this gives a contribution

$$\Pi^{00,\mathrm{scal}}_{\mathrm{mat},1} = -\sqrt{eB}T\frac{2-\sqrt{2}}{4\pi}[-\zeta(1/2)] \quad . \tag{6.29}$$

The other terms are found to be suppressed at high temperatures. Identifying the momentum scale $\kappa^2 = eB$, we thus find in QCD, using for $k_0 = 0$ $\Pi^L_{\mathrm{mat}} = \Pi^{00}_{\mathrm{mat}}$

$$\tilde{\Pi}(x) = x^2\left(\frac{N_c}{3} + \frac{N_f}{6}\right) - x\frac{N_c}{2}\frac{2-\sqrt{2}}{2\pi}|\zeta(1/2)| + \mathcal{O}(1/x) \quad . \tag{6.30}$$

The leading T^2 behavior agrees with Ref. [9], but the coefficient in front of $x = T/\kappa$, approximately $0.1365N_c/2$, does not. The absence of $\ln T$ in Eq. (6.30) indicates that the identification between the magnetic field eB and the momentum scale κ^2 does not work here.

On the other hand, using the gauge invariant Vilkovisky–De Witt effective action indicates that it is the transverse part of the vacuum polarization tensor that governs the renormalization [6, 8]. In this case a cancelation occurs between the leading T^2/κ^2 terms. The coefficient in front of the next leading T/κ term is found to depend on the gauge fixing parameter. However, the Vilkovisky–De Witt effective action speaks in favor of the Landau gauge $\xi = 0$. Recently Elmfors and Kobes [8] used a Braaten–Pisarski resummation scheme [38] to calculate Π^T_{mat} self-consistently in the high-temperature limit. Their result reads in our notation

$$\tilde{\Pi}(x) \simeq -x\frac{N_c}{64}[(3+\xi)^2 + 14] \quad . \tag{6.31}$$

This will give an effective charge increasing with the temperature. The effects of the inclusion of a non-perturbative magnetic mass was also considered in Ref. [8]. In the Landau gauge this did not qualitatively change the asymptotic behavior of the effective charge, whereas it could do so in other gauges with $\xi > 1$. Recently, Sasaki [39] has used the pinch-technique [40] to calculate a gauge-invariant thermal β-function in QCD. It was explicitly shown in various gauges that the thermal β-function is invariant, and

the result agrees with the one obtained in the background field formalism in the Feynman gauge $\xi = 1$. No Braaten–Pisarski resummation was performed in this case, but we have found no reason why Eq. (6.31) with $\xi = 1$ should not be the resummed pinch technique result. We have no answer why the Vilkovisky–De Witt effective action, on general grounds supposed to be gauge-invariant, gives a different result from the pinch technique, explicitly shown to give equal results in various gauges.

6.4 Inclusion of the Thermal Gluon Mass

It is well-known that the solution of some infra-red problems in thermal field theories requires a resummation of the dominant diagrams to all orders in perturbation theory [38]. The result is concluded in the elegant "Hard Thermal Loop Effective Action". Here we are primarily interested in regularizing the infra-red behavior. We shall therefore only take the thermal gluon mass into account. The derivation performed here will only be heuristic. Considering the high temperature limit, we shall use the well-known result [38] for $B = 0$

$$m_G^2 \equiv \Pi^T(k^2 = 0) \simeq \left(N_c + \frac{N_f}{2}\right) g^2 \frac{T^2}{9} \qquad , \tag{6.32}$$

that must be the leading contribution also for $T^2 \gg gB$. This is the on-shell self-energy for transverse gluons only, corresponding to the polarizations $s = \pm 1$ for the charged W bosons. We thus add and subtract a mass term $m_G^2/2 \sum_a E_a^\nu E_\nu^a$ to the $SU(N_c)$ Lagrangian in Eq. (6.1). The subtracted mass term is necessary in order not to change the original Lagrangian, that would ruin gauge invariance. It is treated as a counter term, in order that the hard thermal loops are not counted twice. The resummed transverse gluon self-energy is for example $\Pi^T_{\text{res}} = \Pi^T - m_G^2$. The contributions from this counter term are calculated in Appendix B. Rewritten in terms of W, the resummed Lagrangian corresponds to Eq. (4.3) with $m^2 = m_G^2$. Therefore, we may immediately use the results from massive vector QED, but only with two polarizations ($s = \pm 1$) similar to the mass-less case, since a dynamical mass term cannot change the number of degrees of freedom. Furthermore, we shall assume the thermal mass to be so large that the instability is removed, i.e. $m_G^2 > gB$. With these assumptions we may immediately use our previous results in the high temperature limit to find

$$\begin{aligned} \mathcal{L}^{\text{QCD}}_{\text{mat},1} + \mathcal{L}^{\text{QCD}}_{\text{vac}} \;\; &\simeq \;\; -\frac{1}{2}\hat{\chi}^{\text{QCD}}(gB)^2 \ln\frac{T^2}{\lambda^2} - 2\frac{T}{m_G}\frac{N_c}{2}\frac{(gB)^2}{48\pi^2} \\ &\quad + \sqrt{\frac{N_c}{2}}\frac{gBT}{2\pi}\left(\sqrt{m_G^2 + \sqrt{\frac{N_c}{2}}gB} - \sqrt{m_G^2 - \sqrt{\frac{N_c}{2}}gB}\right) \quad , \end{aligned} \tag{6.33}$$

where g and B denote the charge and field renormalized at the momentum scale λ, respectively. Actually we should here also have included the thermal quark mass $m_q^2 =$

$[(N_c^2-1)/2N_c]g^2T^2$, but this gives no relevant effects. Notice that expanding according to $gB < m_G^2$ in the last term of Eq. (6.33) will only give even powers of

$$\left(\frac{\sqrt{\frac{N_c}{2}}gB}{m_G^2}\right)^{2n} = \left(\frac{N_c}{2}\frac{(gB)^2}{m_G^4}\right)^n \qquad , \tag{6.34}$$

so that the result is group invariant to any order. The correct approach would be to use $e_{\rm vec}^4 = g^4/2\,{\rm Tr}\,[T_{\rm vec}^{a'}T_{\rm vec}^{a'}T_{\rm vec}^{a'}T_{\rm vec}^{a'}]$ etc. for higher powers. Here we shall only consider the leading behavior for $m_G^2 \gg gB$. The contribution from the counter terms is then

$$\mathcal{L}_{\rm mat,1}^{\rm QCD,(c)} + \mathcal{L}_{\rm vac}^{\rm QCD,(c)} \simeq \frac{1}{2}\hat{\chi}^{\rm QCD}(gB)^2\left[I_1\left(\frac{m_G}{T}\right)+1\right] \qquad . \tag{6.35}$$

We may now perform the derivative with respect to B to find the magnetization

$$\frac{M^{\rm QCD}}{g^2B} \simeq -\hat{\chi}^{\rm QCD}\left[\ln\frac{T^2}{\lambda^2} - I_1\left(\frac{m_G}{T}\right) - 1\right] + \frac{N_c}{2}\frac{T}{m_G}\frac{12-1}{12\pi} \qquad . \tag{6.36}$$

We have here written "$12-1$" in the last term to indicate that this coefficient enters exactly as $\hat{\chi}^{\rm vec} = 2\hat{\chi}^{\rm scal} + \Delta\hat{\chi}^{\rm vec}$. In terms of the charge renormalized at momentum scale λ_0, we now find the effective temperature dependent charge

$$\frac{1}{g_{\rm eff}^2(T,gB)} \simeq \frac{1}{g^2(\lambda_0)} + \hat{\chi}^{\rm QCD}\left[\ln\frac{T^2}{\lambda_0^2} - I_1\left(\frac{m_G}{T}\right) - 1\right] - \frac{N_c}{2}\frac{T}{m_G}\frac{11}{12\pi} \qquad . \tag{6.37}$$

Notice that $m_G \propto T$, so that the only T dependence is in the logarithm. The other constant terms may be renormalized away. The effective charge obtained with a thermal mass regularizing the instability is thus decreasing logarithmically in accordance to the vacuum RGE with $\lambda = T$. This was the original assumption by Collins and Perry [11], on the formation of a quark–gluon plasma (a notion later coined) as a result of asymptotic freedom. Also notice that the result follows provided that the thermal mass is $m_\beta \propto T$, and $m_\beta^2 > gB$, regardless of the explicit form of m_β. We could therefore have used the electric mass $m_E^2 = \Pi^L(k_0=0,\mathbf{k}) = 3m_G^2$, or the (nonperturbative) magnetic mass $m_M^2 = \Pi^T(k_0=0,\mathbf{k}) = (cg^2T)^2$. The thermal mass used here follows also in the uniform limit $\Pi^T(k_0,\mathbf{k}=\mathbf{0}) = m_G^2$. We believe that it is the thermal gluon mass used here that is relevant for the physical polarizations of the gluons.

7 Discussion

In this paper we have calculated the effective charge of different gauge theories in vacuum and in a thermal environment. In the vacuum case the effective charge is related to the magnetic susceptibility. The general criteria for (anti-) screening, is that the effective

charge is (increasing) decreasing with the distance or the inverse momentum scale. If the reference charge that we are comparing our effective charge with is taken as the bare charge, measured at infinite energy, the effective charge is always smaller than the bare charge in spinor and scalar QED. This corresponds to screening, and is related to bare diamagnetism. In QCD, on the other hand, the effective charge is always greater than the bare charge. This implies anti-screening, asymptotic freedom and bare paramagnetism. However, in spinor and scalar QED the reference charge is customary taken as the classical charge, measured in the infinite wave-length limit. In this case the effective charge is always greater than the reference charge, and the corresponding vacuum magnetization will show a paramagnetic behavior. These theories do anyhow exhibit screening since the effective charge is increasing with the energy scale. In QCD there is no such natural scale at which to define the effective charge, since the charge is becoming infinitely large in the long wavelength limit. The magnitude of the effective charge compared to the reference charge , and thereby also the sign of the magnetic susceptibility, will therefore depend on the relative magnitude between the scales at which the reference charge and the effective charge are measured.

However, some caution is required in the interpretation of asymptotic freedom in QCD in terms of anti-screening in a dielectricum. In the presence of quarks and gluons, *external fields are screened* also in QCD, see e.g. Ref. [41]. Chromo-electric fields are screened with the electric (Debye) mass $m_E^2 \simeq g^2[(N_c + N_f/2)T^2/3 + \sum_f \mu_f^2/(2\pi^2)]$. To cure some infrared problems also magnetic fields are believed to be screened with a non-perturbatively generated magnetic mass $m_M^2 = \mathcal{O}(g^4T^2)$. We believe that this could be related to the condensate removing the tachyonic mode, as suggested already by Cornwall [40], but this needs to be investigated further. We have here found that the anti-screening is caused by the large spin magnetic moment of the gluons that themselves carry the color charge, in terms of the W field in Eq. (6.6). We may view the anti-screening as an effect of dispersion of the color charge [41]. For example, a static blue (B) quark may become red (R) by emitting a B$\bar{\text{R}}$ gluon. This will effectively distribute the blue charge over a volume $\simeq r^3$. When investigating the charge in a volume $\lambda^{-3} \ll r^3$, only a small fraction of the net blue charge is found.

In the thermal case we mainly focus on the effective charge in QCD. There are several advantages in this approach of calculating the effective charge from the effective Lagrangian in a back-ground magnetic field. Background field gauge-invariance is maintained, even though we only consider one particular choice of gauge here. This gauge-invariance implies that the product of the coupling and the background field are invariant under renormalization. The quark–gluon, three gluon and four gluon couplings are thus renormalized in the same way, and kept equal. Summing over physical degrees of freedom only, in the vacuum energy and the free energy, there is no need to introduce ghosts. The running coupling

obtained from the RGE is a function of scaled momenta $k^\mu \to e^t k^\mu$, but this means that we move off-shell. Most often the effective charge is calculated in the static limit $k^\mu = (0, \kappa\hat{k})$, relevant for the screening of external fields. This means that one is in the deep Euclidean region $k^2 = -\kappa^2$. The running coupling so obtained cannot be ascribed to real particles, only to virtual particles in internal processes. The effective coupling obtained from the effective Lagrangian, on the other hand, is directly related to the interactions of the real particles in the heat and charge-bath. Moreover, the thermal contribution to the effective Lagrangian is related to the free energy. Since physical, measurable quantities are obtained by performing derivatives on the free energy, it must be gauge invariant and cannot depend on the gauge fixing, up to an irrelevant constant.

At fixed magnetic field, we find from Eq. (6.20) the leading behavior of the effective coupling in QCD as a function of the temperature

$$g^2_{\text{eff}}(T, gB) = \frac{g^2_{\text{eff}}(T_0, gB)}{1 + g^2_{\text{eff}}(T_0, gB)\frac{3}{2\pi^2}\left(\frac{N_c}{2}\right)^{3/4}\left[\frac{T}{\sqrt{gB}}\ln\frac{T}{\sqrt{gB}} - \frac{T_0}{\sqrt{gB}}\ln\frac{T_0}{\sqrt{gB}}\right]} \quad . \tag{7.1}$$

As found when explicitly calculating parts of the vacuum polarization tensor, we cannot transform the dependence on the magnetic field, to dependence of momentum through $\kappa^2 \approx eB$. However, we may consider the limit of vanishing magnetic field. In the limit $gB \to 0$ in Eq. (7.1) we find the simple behavior

$$\frac{g^2_{\text{eff}}(T)}{g^2_{\text{eff}}(T_0)} = \frac{T_0}{T} \quad , \tag{7.2}$$

i.e. a coupling *linearly decreasing* with the temperature. The effective charge obtained in the limit of vanishing magnetic field has the advantage of being independent of the direction in color space for the magnetic field, unlike the effective charge in Eq. (7.1) as indicated by the non-analytical behavior in terms of N_c. Notice, however, that the appearance of the $(gB)^{3/2}T\ln(T/\lambda)$ term in the effective Lagrangian, that enforces asymptotic freedom at high temperatures, is solely due to the negligence of the tachyonic mode. If this mode is taken into account, by analytically continue $m^2 \to -i\varepsilon$ in Eq. (5.15), this term disappears. But we are in that case left with an imaginary part in the free energy. Considering the real part only we find in the high temperature limit an *increasing* effective charge

$$g^2_{\text{eff}}(T, gB) = \frac{g^2_{\text{eff}}(T_0, gB)}{1 - g^2_{\text{eff}}(T_0, gB)\frac{3}{4\pi}\left(\frac{N_c}{2}\right)^{3/4}\left[\frac{T}{\sqrt{gB}} - \frac{T_0}{\sqrt{gB}}\right]} \quad . \tag{7.3}$$

Also this effective charge suffers from non-invariance under transformations in the color group $SU(N_c)$, and thus cannot be trusted. In this case we cannot take the limit $gB \to 0$, since this would take us across the Landau pole, at which the coupling is becoming infinitely strong. We obtain qualitatively different results whether the tachyonic mode is neglected, or

the real part of its contribution is taken into account. However, particularly when starting from the free energy, we consider it physically more reasonable to neglect all contributions from the imaginary energy mode, that of course cannot be present in a real situation. The group invariant charge obtained in the field free limit when neglecting the contribution from the tachyonic mode could thus be taken as an indication of the true situation.

A thorough investigation of this tachyonic mode has so far required numerical treatment, and the (small) change in the magnetic field due to the condensate in this mode would also alter the other modes present, and thereby also the effective Lagrangian. This seems to destroy the advantage of simplicity in obtaining the effective charge, using the effective Lagrangian in a magnetic field. However, including a thermal gluon mass may take us out of this dilemma. We find in the high temperature limit when the thermal mass is large enough to completely remove the instability, an effective charge *decreasing according to the vacuum RGE* (4.27) with $\lambda \mapsto T$

$$\frac{1}{g_{\rm eff}^2(T)} = \frac{1}{g_{\rm eff}^2(T_0)} + \hat{\chi}^{\rm QCD} \ln\left(\frac{T}{T_0}\right)^2 \qquad . \tag{7.4}$$

This is what naively has been expected for long [11], but hitherto not obtainable. Obviously, this effective charge is also valid in the absence of a (chromo-) magnetic field. However, the derivation outlined here is merely heuristic, and more rigorous considerations are to be be performed.

If we instead consider the case of large flavor chemical potentials, we may immediately use the expansions in Eq. (5.19) and Eq. (5.23). It is the latter, oscillating contribution that is most interesting. It is oscillating with an amplitude $|M_{\rm osc}|/e^2B \propto \mu/\sqrt{eB}$, but it may be positive as well as negative depending on the ratio $(\mu^2 - m^2)/(2eB)$. The effective coupling thus may increase as well as decrease. However, due to these rapid oscillations we are doubtful to this definition of the effective charge in the limit of large chemical potentials and low temperatures.

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APPENDIX

A High Temperature Expansions

A.1 Massive Vector QED

We shall here find the high temperature limit of

$$\Delta\mathcal{L}^{\text{vec}}_{\text{mat}} = \frac{eB}{\pi^2}\int_0^\infty dp_z p_z^2 \left[\frac{1}{E_{0,-1}}\frac{1}{e^{\beta E_{0,-1}}-1} - \frac{1}{E_{0,0}}\frac{1}{e^{\beta E_{0,0}}-1}\right] \quad . \tag{A.1}$$

Using the expansion of the distribution function

$$\frac{1}{e^{\beta E}-(-1)^{2\sigma}} = \sum_{l=1}^{\infty}(-1)^{2\sigma(l-1)}e^{-\beta El} \quad , \tag{A.2}$$

and the identity

$$\frac{e^{-\beta El}}{\beta El} \equiv \int_0^\infty \frac{dt}{\sqrt{2\pi t}}\exp\left[-\frac{1}{2}\left(\beta^2E^2l^2t+\frac{1}{t}\right)\right] \quad , \tag{A.3}$$

we may perform the Gaussian p_z-integral in Eq. (5.14). We can also identify [42]

$$\int_0^\infty \frac{dt}{t^{\nu+1}}\exp\left[-bt-\frac{c}{t}\right] = 2\left(\frac{b}{c}\right)^{\nu/2}K_\nu(2\sqrt{bc}) \quad , \tag{A.4}$$

where $K_\nu = K_{-\nu}$ is a modified Bessel function. We may now write

$$\Delta\mathcal{L}^{\text{vec}}_{\text{mat}} = \frac{eBT^2}{\pi^2}\left[I\left(\frac{\sqrt{m^2-eB}}{T}\right) - I\left(\frac{\sqrt{m^2+eB}}{T}\right)\right] \quad , \tag{A.5}$$

where

$$I(x) \equiv \sum_{l=1}^{\infty}\frac{1}{l^2}(xl)K_1(xl) \quad . \tag{A.6}$$

Using the derivative property of Bessel functions [43], we find [42]

$$\left(\frac{1}{x}\frac{d}{dx}\right)I(x) = -\sum_{l=1}^{\infty}K_0(xl) = -\left\{\frac{1}{2}\left(\gamma_E+\ln\frac{x}{4\pi}\right)-\frac{\pi}{2x}+\mathcal{O}(x)\right\} \quad . \tag{A.7}$$

This finally gives

$$\begin{aligned}\Delta\mathcal{L}^{\text{vec}}_{\text{mat}} \;=\; & -\frac{(eB)^2}{4\pi^2}\ln\left[\frac{T^2(4\pi)^2}{\sqrt{m^4-(eB)^2}}\right] + \frac{eBT}{2\pi}(\sqrt{m^2+eB}-\sqrt{m^2-eB}) \\ & + \frac{eBm^2}{8\pi^2}\ln\left(\frac{m^2+eB}{m^2-eB}\right) - \frac{(eB)^2}{2\pi^2}\left(\frac{1}{2}-\gamma_E\right) + eBT^2\mathcal{O}\left(\frac{m^2+eB}{T^2}\right)^{3/2}\end{aligned} \tag{A.8}$$

In order for this to be valid also for $eB > m^2$, we must consider the analytical continuation defined by $m^2 \to m^2 - i\varepsilon$. The same result is obtained also if we treat the contribution for $p_z^2 < eB$ separately in analogy to the vacuum case. The result is presented in Eq. (5.15).

A.2 Mass-less Spinor and Scalar QED

In the mass-less case, we have found it necessary to add the vacuum contribution in order to perform a Poisson resummation and find the high temperature behavior. We have not managed to perform this analysis using the cut-off regularization previously utilized. Therefore, we shall in this section instead use a dimensional regularization in $4-2\delta$ dimensions. In order to keep the coupling dimension-less, we substitute $e \mapsto \lambda^\delta e$, where λ is an energy scale. For $E_{n,s} > \mu$ we may use the expansion in Eq. (A.2), and the identity in Eq. (A.3). Performing the summation over n, we find

$$\begin{aligned}\mathcal{L}_{\mathrm{mat}} &= \frac{eB\lambda^\delta}{2\pi}\sum_s \int_0^\infty \frac{dt}{\sqrt{2\pi t}} \sum_{l=1}^{\infty}(-1)^{2\sigma(l-1)}\beta l\cosh(\beta l\mu)\int\left(\frac{dp_z}{2\pi}\right)^{1-2\delta} p_z^2\exp\left(-\frac{1}{2t}\right)\\ &\quad\times\frac{\exp[-\beta^2 l^2 t(p_z^2 - eB\lambda^\delta\gamma s)/2]}{\sinh(\beta^2 l^2 eB\lambda^\delta t/2)} \quad . \end{aligned} \tag{A.9}$$

Integrating over p_z, subtracting the $B=0$ part $\mathcal{L}_{\mathrm{mat},0}$ and substituting $u = \beta^2 l^2 eB\lambda^\delta t/2$, we find

$$\begin{aligned}\mathcal{L}_{\mathrm{mat},1} &= \frac{(eB)^2}{8\pi^2}\left(\frac{4\pi\lambda^2}{eB}\right)^\delta \sum_{l=1}^{\infty}(-1)^{2\sigma(l-1)}\cosh(\beta l\mu)\int_0^\infty \frac{du}{u^{3-\delta}}\exp\left(-\frac{\beta^2 l^2 eB\lambda^\delta}{4u}\right)\\ &\quad\times\sum_s\left\{\frac{u\exp(\gamma s u)}{\sinh(u)} - 1\right\} \quad . \end{aligned} \tag{A.10}$$

We now wish to perform a Poisson resummation, and must therefore add the $l=0$ term to extend the summation to $\sum_{l=-\infty}^{\infty}$. This term with $l=0$ is the bare vacuum Lagrangian, regularized in $4-2\delta$ dimensions, cf. Eq. (4.12). Let us now for simplicity consider the case of a neutral plasma, $\mu = 0$. Again the vector case needs special care, so here we first consider only spinor and scalar QED with $\sigma = 1/2$ and $\sigma = 0$, respectively. Performing the Poisson resummation we find

$$\begin{aligned}\mathcal{L}_{\mathrm{vac}} + \mathcal{L}_{\mathrm{mat},1} &= (eB)^2\frac{2T\sqrt{\pi}}{\sqrt{eB\lambda^\delta}}\left(\frac{4\pi\lambda^2}{eB}\right)^\delta \sum_{k=-\infty}^{\infty}\int_0^\infty \frac{du}{u^{5/2-\delta}}\exp\left[-\frac{4\pi^2T^2}{eB\lambda^\delta}(k+\sigma)^2 u\right]\\ &\quad\times\frac{(-1)^{2\sigma}}{16\pi^2}\sum_s\left\{\frac{u\exp(\gamma s u)}{\sinh[u]} - 1\right\} \quad . \end{aligned} \tag{A.11}$$

The term with $k=0$ for bosons ($\sigma = 0$) gives a contribution to $\mathcal{L}_{\mathrm{mat}}^{\mathrm{scal}}$, finite for $\delta = 0$

$$\frac{(eB)^{3/2}T}{8\pi^{3/2}}\int_0^\infty \frac{du}{u^{5/2}}\left\{\frac{u}{\sinh[u]} - 1\right\} = \frac{(eB)^{3/2}T}{8\pi^{3/2}}(2^{5/2}-4)\sqrt{\pi}\zeta(-\frac{1}{2}) \quad , \tag{A.12}$$

where $\zeta(-1/2) \simeq -0.207886$, and we have used $\Gamma(-1/2) = -2\sqrt{\pi}$. We have here identified Riemann's zeta function $\zeta(z)$ from an integral representation in Ref. [42] that is valid for $\mathrm{Re}\,(z) > 1$. However the subtracted "1" on the left hand side of Eq. (A.12) is regularizing

this expression (the integral is manifestly convergent), and we have numerically checked the equality. For the other terms in the sum, we sum over s and expand [42]

$$\frac{(-1)^{2\sigma}}{8\pi^2}\left\{\frac{u\cosh(\gamma\sigma u)}{\sinh[u]}-1\right\}=\frac{1}{2}\hat{\chi}u^2-\frac{1}{8\pi^2}\sum_{n=2}^{\infty}\frac{2^{2n}-2(1-\gamma\sigma)}{(2n)!}B_{2n}u^{2n} \quad , \tag{A.13}$$

for $\gamma\sigma = 0,\ 1$. We may then integrate over u. Due to the alternating sign of the Bernoulli numbers B_{2n}, the modulus of the sum is smaller than the modulus of the first neglected term. For $\nu \geq 2$ we may let $\delta = 0$, and find the contribution

$$-\frac{1}{8\pi^2}\frac{2^{2n}-2(1-\gamma\sigma)}{(2n)!}B_{2n}(eB)^2\left(\frac{eB}{4\pi^2T^2}\right)^{2(\nu-1)}\frac{\Gamma(2\nu-3/2)}{\sqrt{\pi}}\zeta(4\nu-3,1-\sigma) \quad , \tag{A.14}$$

that is suppressed for large T. From the first term in the expansion of Eq. (A.13) we obtain a contribution

$$\frac{1}{2}\hat{\chi}(eB)^2\left(\frac{\lambda^2}{\pi T^2}\right)^{\delta}\frac{\Gamma(\frac{1}{2}+\delta)}{\sqrt{\pi}}{\sum_k}'\frac{1}{|k+\sigma|^{1+2\delta}} \quad , \tag{A.15}$$

where $\sum_k'$ means that the term with $k = 0$ should be excluded for $\sigma = 0$. We can now identify a generalized Riemann's ζ-function [42]

$${\sum_k}'\frac{1}{|k+\sigma|^{1+2\delta}}\equiv 2\sum_{k=0}^{\infty}\frac{1}{[k+(1-\sigma)]^{1+2\delta}}\equiv 2\zeta(1+2\delta,1-\sigma) \quad . \tag{A.16}$$

As $\delta \to 0$ we have the expansion [42]

$$\zeta(1+2\delta,1-\sigma)=\frac{1}{2\delta}+\gamma+2\sigma\ln 2+\mathcal{O}(\delta) \quad , \tag{A.17}$$

for $\sigma = 0, 1/2$. We now expand $\Gamma(\frac{1}{2}+\delta) = \sqrt{\pi}+\mathcal{O}(\delta)$, $(\lambda^2/\pi T^2)^\delta = 1-\delta\ln(\pi T^2/\lambda^2)+\mathcal{O}(\delta)$, perform a renormalization to absorb the divergent term $1/\delta$ and all terms $\propto (eB)^2$ in the bare coupling, and then let δ vanish.

A.3 Mass-less Vector QED

In this case we cannot employ a technique similar to Eq. (A.12), since the integral would become divergent due to the $\exp(\gamma s u)$. Here we shall neglect the tachyonic mode and start integrating at $p_z^2 = eB - m^2$ in the contribution from the lowest Landau level. Since the lowest energy then is vanishing, only $\mu = 0$ is possible in this case. If we have a finite charge density it must therefore reside in a Bose-Einstein condensate in the zero energy mode. When the tachyonic mode is not included, integrations by part will cause non-vanishing surface terms. We shall here start from the free energy in Eq. (5.6), that we believe is most physical. We then have

$$\Delta\mathcal{L}_{\text{mat}}^{\text{vec}}=-\frac{eBT}{\pi^2}\left\{\int_{\sqrt{eB}}^{\infty}dp_z\ln[1-e^{-\beta E_{0,1}}]-\int_0^{\infty}dp_z\ln[1-e^{-\beta E_{0,0}}]\right\} \quad . \tag{A.18}$$

Substitute $x = \beta E_{0,s}$, and split the integral at $1 \gg x_0 \gg \sqrt{eB}/T$. Expanding for $x > x_0$

$$\frac{1}{\sqrt{x^2 + eB/T^2}} - \frac{1}{\sqrt{x^2 - eB/T^2}} = \frac{eB}{T^2}\frac{1}{x^3} + \mathcal{O}\left(\frac{eB}{T^2}\right)^3 \quad . \tag{A.19}$$

The remaining integral is convergent, so this will only produce an irrelevant $\mathcal{O}(eB)^2$ term, and terms suppressed at large T. For $x < x_0$ we instead expand

$$\ln[1 - e^{-x}] = \ln x - \frac{x}{2} + \mathcal{O}(x^2) \quad . \tag{A.20}$$

We may now perform the integrals over x to find Eq. (5.18).

B Thermal Gluon Mass

We shall here briefly outline the results from the inclusion of a thermal mass m_β. Let us start without the matter contribution. Considering one-loop effects only, we may integrate out the particles in the generating functional of Green's functions. Adding and subtracting the term corresponding to the thermal mass, we find in the simplest case of a mass-less (i.e. without the thermal mass) scalar field

$$\int d^4x \mathcal{L}_{\text{eff}} = \int d^4x \left(-\frac{1}{4}F^2\right) + i \,\text{Tr}\, \ln[i(-D^2 - m_\beta^2 + m_\beta^2)] \quad . \tag{B.1}$$

We now wish to define our perturbative expansion in terms of the massive field. Let us therefore split the logarithm and the effective Lagrangian according to

$$\int d^4x \left(\mathcal{L}_{\text{vac}} + \mathcal{L}_{\text{vac}}^{(c)}\right) = i \,\text{Tr}\, \ln[i(-D^2 - m_\beta^2)] + i \,\text{Tr}\, \ln\left[1 + \frac{m_\beta^2}{-D^2 - m_\beta^2}\right] \quad . \tag{B.2}$$

Performing the m_β^2 derivative on the first term, we recognize the propagator $(-D^2 - m_\beta^2)^{-1}$. This may then be generalized to arbitrary spin, resulting in Eq. (4.5) for the case of a magnetic field considered here. Expanding the second logarithm in Eq. (B.2) to leading order, we again find the trace of the propagator. Similar generalizations to arbitrary spin give

$$\mathcal{L}_{\text{vac}}^{(c)} \simeq m_\beta^2\, i(-1)^{2\sigma} \frac{eB}{(2\pi)^3} \sum_s \sum_{n=0}^{\infty} \int d\omega dp_z \frac{1}{w^2 - E_{n,s}^2 + i\varepsilon} \quad . \tag{B.3}$$

Integrating over ω, we may then use Eq. (4.11). In this counter term we shall encounter no divergences, so we may immediately let $1/\Lambda = 0$. We may then perform the Gaussian integral in p_z, and sum the infinite geometrical series in n. Substituting $t = x^2$, and subtracting the contribution for $B = 0$, we find

$$\mathcal{L}_{\text{vac}}^{(c)} \simeq m_\beta^2 \frac{(-1)^{2\sigma}}{16\pi^2} \sum_s \int_0^\infty \frac{dt}{t^2} \exp(-m_\beta^2 t) \left\{\frac{eBt}{\sinh(eBt)} \exp(eBt\gamma s) - 1\right\} \quad . \tag{B.4}$$

Expanding for $eBt \lesssim eB/m_\beta^2 \ll 1$, we find

$$\mathcal{L}_{\rm vac}^{(c)} \simeq \frac{1}{2}\hat{\chi}(eB)^2 \left[1+\mathcal{O}\left(\frac{eB}{m_\beta^2}\right)^2\right] \quad . \tag{B.5}$$

Let us now consider the thermal contribution. Similar generalizations as in the vacuum case leads for $\mu = 0$ to

$$\mathcal{L}_{\rm mat}^{(c)} \simeq m_\beta^2 \frac{eB}{(2\pi)^2} \sum_s \sum_{n=0}^{\infty} \int dp_z \frac{1}{E_{n,s}} \frac{1}{e^{\beta E_{n,s}} - (-1)^{2\sigma}} \quad . \tag{B.6}$$

Expanding the distribution function according to Eq. (A.2), and using Eq. (A.3), we may again perform the Gaussian integral in p_z, and sum the infinite geometrical series in n. Subtracting the contribution for $B = 0$, we find

$$\begin{aligned} \mathcal{L}_{\rm mat,1}^{(c)} &= \frac{(-1)^{2\sigma}}{8\pi^2} \sum_s \sum_{l=1}^{\infty} (-1)^{2\sigma l} \frac{1}{\frac{1}{2}\beta^2 m_\beta^2 l^2} \int_0^\infty \frac{dt}{t^2} \exp\left[-\frac{1}{2}\beta^2 m_\beta^2 l^2 t - \frac{1}{2t}\right] \\ &\quad \times \left\{ \frac{\frac{1}{2}\beta^2 l^2 eBt}{\sinh(\frac{1}{2}\beta^2 l^2 eBt)} \exp(\frac{1}{2}\beta^2 l^2 eB\gamma st) - 1 \right\} \quad . \end{aligned} \tag{B.7}$$

We may again expand according to $eB\beta^2 l^2 t/2 \lesssim eB/m_\beta^2 \ll 1$. Using Eq. (A.4) the result reads

$$\mathcal{L}_{\rm mat,1}^{(c)} \simeq \frac{1}{2}\hat{\chi}(eB)^2 \left[I_\sigma(m_\beta/T) + \mathcal{O}(eB/m_\beta^2)\right] \quad , \tag{B.8}$$

where we have defined

$$I_\sigma(x) \equiv 2\sum_{l=1}^{\infty} (-1)^{2\sigma l} x l K_1(xl) \quad . \tag{B.9}$$

For small z, $zK_1(z) \approx 1$, and for large z, $zK_1(z) \approx \sqrt{\pi z/2}\exp(-z)$ [43], so that the sum is convergent. Now, to leading order, $m_\beta = m_G \propto T$, so $I_\sigma(m_\beta/T)$ is T-independent. For completeness we may also write down the field independent contribution

$$\mathcal{L}_{\rm mat,0}(c) = m^4 \frac{(-1)^{2\sigma}}{2\pi^2} \sum_s \sum_{l=1}^{\infty} \frac{(-1)^{2\sigma l}}{\beta m l} K_1(\beta m l) \quad . \tag{B.10}$$

We have found no analytical expression for this series, but its leading behavior is $\mathcal{O}(m_\beta^2 T^2)$, i.e. $\mathcal{O}(g^2 T^4)$, sub-leading for small couplings. Of course, this field independent part is irrelevant for the effective coupling.

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